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d'Alembert–Lagrange analytical dynamics for nonholonomic systems

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The d'Alembert–Lagrange principle (DLP) is designed primarily for dynamical systems under ideal geometric constraints. Although it can also cover linear-velocity constraints, its application to nonlinear kinematic constraints has so far remained elusive, mainly because there is no clear method whereby the set of linear conditions that restrict the virtual displacements can be easily extracted from the equations of constraint. On recognition that the commutation rule traditionally accepted for velocity displacements in Lagrangian dynamics implies displaced states that do not satisfy the kinematic constraints, we show how the property of possible displaced states can be utilized *ab initio* so as to provide an appropriate set of linear auxiliary conditions on the displacements, which can be adjoined via Lagrange's multipliers to the d'Alembert–Lagrange equation to yield the equations of state, and also new transpositional relations for nonholonomic systems. The equations of state so obtained for systems under general nonlinear velocity and acceleration constraints are shown to be identical with those derived (in Appendix A) from the quite different Gauss principle. The present advance therefore solves a long outstanding problem on the application of DLP to ideal nonholonomic systems and, as an aside, provides validity to axioms as the Chetaev rule, previously left theoretically unjustified. A more general transpositional form of the Boltzmann–Hamel equation is also obtained. © *2011 American Institute of Physics*. [doi:10.1063/1.3559128]

I. INTRODUCTION

The past decade has witnessed continued growing interest[1–9] in theoretical analysis of nonholonomic systems. Modern developments in robotics,[10] vehicular control, sensors, feedback control, servomechanisms, cruise controls, and other advanced technologies make possible interesting systems operating under general nonholonomic constraints. Graduate texts[11] on analytical dynamics primarily deal with the 222-year-old fundamental d'Alembert–Lagrange principle[12,13] (DLP) of 1788, which involves virtual displacements $\delta\mathbf{r}$ to the particle's position $\mathbf{r}(t)$ with constraints held frozen during the displacement and which yields the familiar Lagrange's equations of state. Although Lagrange[12] designed it with only geometric constraints in mind, DLP can also be applied[14–18] to kinematic constraints, linear in velocity, whose very existence even Lagrange at that time did not anticipate. Lagrange assumed that independent coordinates could always be chosen for any system once the constraints were acknowledged (embedded). Hertz,[19] over 100 years later, was the first to recognize in 1894 the essential difference between geometric (holonomic) constraints on the configuration and nonintegrable kinematic (nonholonomic) constraints which directly restrict the velocities/accelerations of the state. Gauss[20] in 1829 had already provided a very different principle of Least Constraint,[14–16,20–22] based on virtual displacements to the acceleration alone, keeping the state $(\mathbf{r}, \dot{\mathbf{r}})$ fixed. This principle provides (as in Appendix A) equations of state for both holonomic *and* nonholonomic systems. In this paper, we wish to extend, with full theoretical justification, the capability of DLP to treat general nonholonomic systems.

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There are now essentially four approaches to the dynamics of nonholonomic systems. The first, analytical dynamics as followed here, is based on the traditional DLP and on Hamilton's constrained principle, which follows from DLP and is valid[6,15–17] only for holonomic and integrable constraints. By analogy with DLP, Gibbs[23] and then Jourdain[24] introduced methods based, respectively, on virtual displacements at time t to the acceleration keeping the state $\{\mathbf{r},\dot{\mathbf{r}}\}$ fixed and to the velocity keeping the configuration $\mathbf{r}$ fixed. These methods were originally deduced[15,23,24] by comparison with similar DLP conditions on the virtual displacements δq_j of the generalized coordinates q_j for *geometric and linear-velocity constraints*, respectively, but have recently been shown[25] to also cover general nonlinear kinematic constraints. A second approach, which is the most comprehensive in coverage and quite different from DLP, is based on the Gauss principle[20] and on the resulting Gibbs–Appell equations[15,23,26–29] of motion. Geometric analysis,[30–36] a third approach, reformulates the essential principles of analytical dynamics in geometric terms, using tools from modern differential (Riemannian) geometry, Lie groups, manifolds, jet bundles, and topology. It offers a different perspective and language to the above traditional methods and has been applied mainly to systems that can be also covered by DLP.

Unlike geometric and linear-velocity constraints, the basic unsurmountable difficulty with the implementation of DLP to nonlinear velocity constraints was that there was no obvious way of extracting from the kinematic constraint equations, the set of conditions, linear in $\delta\mathbf{r}$, needed for adjoining them via Lagrange's multipliers to DLP. The comments of Lanczos[21] on DLP and nonholonomic systems pertain *only* to nonintegrable linear-velocity constraints. Various postulates, such as Chetaev's rule[37] for DLP application, a reinterpreted variational principle,[38] and vakonomic mechanics,[39–41] based on the premise that Hamilton's constrained principle is valid even for nonholonomic systems, were therefore advocated for nonlinear velocity constraints. These prescriptions have, however, remained without any basic theoretical justification and have therefore provoked some criticism[9,42] and continuing study.[4,43–48]

In this paper, we illustrate from first principles how systems under general velocity and acceleration constraints may indeed be analyzed by DLP with full justification. The present development recognizes that the commutation rule, $\delta\dot{q}_j = d(\delta q_j)/dt$, conventionally/tacitly assumed in DLP analytical dynamics[11,12,14,15,41] and in Hamilton's constrained principle, is unnecessarily restrictive when it comes to dealing with how the dependent velocity displacements $\delta\dot{q}_j$ in nonholonomic systems are related to the configuration displacements δq_j. Although successful for integrable systems, the commutation rule proves remarkably unfruitful for nonintegrable systems, because, in contrast to integrable systems, its use implies that the displaced states no longer satisfy[6,16,17] the nonholonomic constraints, thereby inhibiting further progress in finding the constraint requirements on the displacements. From the desired property that the displaced states $(q+\delta q, \dot{q}+\delta\dot{q})$ are made possible by obeying the kinematic constraints, we will derive the key set of linear auxiliary conditions on the displacements δq_j needed for adjoining them directly via Lagrange's multipliers to the d'Alembert–Lagrange equation to effect solution. The basic problem of implementation of DLP to general nonholonomic systems is then resolved by the present theory. New transpositional relations, which show how to calculate the constrained velocity or acceleration displacements $\delta\dot{q}_j$ or $\delta\ddot{q}_j$ from δq_j or $\delta\dot{q}_j$ for nonintegrable systems, and which are a consequence of the property of possible states, are then derived.

Historically,[49] there were two viewpoints on transpositional relations. The first maintained that the commutation rule was satisfied by all true generalized coordinates q_j for holonomic and nonholonomic systems. The second maintained that the commutation rule held only for the dependent coordinates of nonholonomic systems and that appropriate rules for the dependent coordinates should be obtained from the constraint equations. Previous work[16,22,49,50] was mainly confined, however, to transpositional relations involving the displacements in the true velocities $\dot{q}_j$ and quasivelocities $\dot{\theta}_j$ for use in the derivation of the Boltzmann–Hamel equation,[14,16,49] which is free from Lagrange's multipliers because the linear-velocity constraints considered are chosen as quasivelocities which can then be naturally embedded. More recently, a transpositional rule at variance with the Chetaev rule was derived,[48] for a particular restricted class of nonholonomic constraint to yield quite complicated Lagrange's equations without Lagrange's multipliers. The present approach, which is quite different from all previous work,[16,22,48–50] is based on obtaining the constraint conditions on δq_j from the property of possible displaced states which permits implementation of DLP to provide equations of state in a form suitable for direct comparison with those derived (in Appendix A) from the different principle of Gauss. It then provides a consistent set of new transpositional relations for ideal nonholonomic systems.

We adopt a generalized-coordinate (rather than Newtonian) representation because the essential argument, although in abbreviated notation, remains more physically transparent. Because transpositional relations can appear awkward and unnecessarily cumbersome, they are expressed here in compact form via Lagrangian derivatives of the constraint equations. We shall first briefly recall the known results of the d'Alembert–Lagrange principle, in order to provide a foundation for the present development, to expose various inherent subtleties and to highlight traditional rules, whose strict adherence may have necessarily impeded previous implementation of DLP to dynamical systems under nonlinear velocity and acceleration constraints.

II. D'ALEMBERT–LAGRANGE PRINCIPLE

The classical state, the representative point $(q,\dot{q})$ of a system at time t of N-particles with Lagrangian L and $n = 3N$ generalized coordinates q_j is, in principle, determined by the solution of

$$L_j \overset{def}{\equiv} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = Q_j^{NP} + Q_j^C \quad (j = 1,2,\ldots,n = 3N) \tag{2.1}$$

obtained by setting the Lagrangian derivative L_j equal to the known applied nonpotential forces Q_j^{NP} plus the unknown forces Q_j^C which constrain the system. The Lagrangian $L(q,\dot{q},t)$ in (2.1) is unconstrained, being written in terms of the $n(= 3N)$ generalized coordinates q_j and n velocities $\dot{q}_j$ for the unconstrained system. Because the constraint forces Q_j^C are generally unknown, (2.1) cannot be solved, except under the special circumstance when Q_j^C are "ideal," in that the summed virtual work $Q_j^C \delta q_j$ done in the virtual displacement $\delta q(t)$ from the unknown physical configuration $q(t)$ vanishes. The constrained system then evolves with time in such a manner that the summed projections,

$$\boxed{(L_j - Q_j^{NP})\,\delta q_j = Q_j^C \delta q_j = 0 \quad (\text{d'Alembert–Lagrange principle})}, \tag{2.2}$$

of each Q_j^C onto δq_j along the q-surface are zero. This is the DLP, a fundamental principle of analytical dynamics established by Lagrange[12] and based on the J. Bernoulli principle of virtual work in statics and the d'Alembert principle[13] for a single rigid body. The summation convention for repeated indices $(j = 1,2,\ldots,n = 3N)$ is adopted throughout. The coefficient $(L_j - Q_j^{NP})$ of δq_j is the projection $(m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i)(\partial \mathbf{r}_i/\partial q_j)$ of Newton's equations summed over all N-particles at positions $\mathbf{r}_i$ onto the various tangent vectors $\hat{q}_j \equiv (\partial \mathbf{r}_i/\partial q_j)$ along direction of increasing q_j alone on the multisurface $q = \{q_j\}$. The Newtonian equivalent of (2.2) is then $(m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i).\,\delta\mathbf{r}_i = 0$, where the forces $\mathbf{F}_i$ exclude the constraint forces. The principle (2.2) is therefore limited to these "workless" constraints Q_j^C, called "ideal" or "perfect" and applies to a wide class of problems which can be solved without direct knowledge of the forces actuating the constraints. The $n = 3N - \delta q_j$'s, in general, are not all independent of each other but are linked by the unknown constraints acting on the system, so that the coefficient $(L_j - Q_j^{NP})$ of each δq_j in (2.2) cannot be arbitrarily set to zero.

Although (2.2) is, in principle, valid for all ideal constraints, its application has been limited to geometric and linear-velocity constraints because the relations restricting the displacements δq_j are then easy to determine in the linear form required for "adjoining" them to the already linear set (2.2) via Lagrange's multiplier method. Application to general nonholonomic systems has so far remained elusive because the auxiliary conditions on the displacements have been impossible to determine in a manner consistent with the conventional commutation rule, $\delta\dot{q}_j = d(\delta q_j/dt)$, traditionally accepted for calculation of the velocity displacements $\delta\dot{q}_j$. However, we shall show how application of (2.2) to general kinematic constraints can indeed be accomplished from the property of possible displaced states, which in turn allows determination of the displacement conditions and construction of a consistent set of transpositional relations to be established between $\delta\dot{q}_j$ and $d(\delta q_j/dt)$ for velocity constraints and between $\delta\ddot{q}_j$ and $d(\delta\dot{q}_j/dt)$ for acceleration constraints. The δq_j in (2.2) will then be compatible with the constraints.

A. Background: Geometric constraints embedded and adjoined

When the constraints are described by geometric equations,

$$f_k(q,t) = 0 \quad (k = 1,2,\ldots,c), \tag{2.3}$$

in configuration space, they are holonomic because they impose restrictions on the possible configurations of the system (and consequently on velocities and accelerations). Often (2.3), at the outset, can be utilized to reduce the number $n = 3N$ of generalized coordinates to the least number $m = (n - c)$ of independent coordinates, which is the actual number of degrees of freedom. The remaining m-infinitesimal displacements δq_j in (2.2) are then all independent—free from the constraints—and are arbitrary. For these holonomic constraints so "embedded" within the reduced Lagrangian L, expressed in terms of the m-independent coordinates and associated velocities, (2.2) yields

$$\boxed{L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} \quad (j = 1,2,\ldots,m), \quad \text{(embedded EOS)}}, \tag{2.4}$$

which are the Lagrange's equations of state (EOS), to be solved for the explicit time t-dependence of the m-independent coordinates $q_j(t)$.

Possible states $(q,\dot{q})$ satisfy the velocity form,

$$\dot{f}_k(q,t) = \left(\frac{\partial f_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial f_k}{\partial t} = 0, \tag{2.5}$$

of (2.3) while the actual physical state must, in addition, satisfy (2.4). Because virtual displacements δq coincide with possible displacements dq in the limit of frozen constraints[18] when $(\partial f_k/\partial t)dt = 0$, they therefore satisfy the linear set of conditions,

$$\delta f_k = \left(\frac{\partial f_k}{\partial q_j}\right)\delta q_j = 0 \quad (k = 1,2,\ldots,c), \tag{2.6}$$

suitable for adjoining directly to (2.2), which is also linear in δq_j. The virtual displacement δq takes the system from one possible configuration, where $f_k(q,t) = 0$, to another possible configuration where the constraint conditions $f_k(q + \delta q,t) = 0$ are again satisfied. The displaced state $(q + \delta q,\dot{q} + \delta\dot{q})$ is also possible because $\dot{f}_k(q + \delta q,\dot{q} + \delta\dot{q},t) = 0$, as shown in Sec. II C. For a given f_k, (2.6), which involves dependent and independent δq_j, may be used directly in order to reduce (2.2) to a summation only over m-independent δq_j and the particular EOS can then be obtained. For general f_k, the conditions (2.6) can be formally adjoined to (2.2) by the Lagrange multiplier method which, in turn, provides the extended principles

$$\left[L_j - Q_j^{NP} - \lambda_k\left(\frac{\partial f_k}{\partial q_j}\right)\right]\delta q_j = 0 \quad \text{(adjoined DLP)} \quad (j = 1,2,\ldots,n) \tag{2.7}$$

for the state and

$$\left[Q_j^C - \lambda_k\left(\frac{\partial f_k}{\partial q_j}\right)\right]\delta q_j = 0 \quad (j = 1,2,\ldots,n) \tag{2.8}$$

for the constraint forces, where all the δq_j can now be all regarded as free (independent) and arbitrary. Then (2.7) and (2.8) readily yield both the familiar n-equations of state

$$\boxed{L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \lambda_k\left(\frac{\partial f_k}{\partial q_j}\right) \quad (j = 1,2,\ldots,n) \quad \text{(adjoined EOS)}}, \tag{2.9}$$

where the unconstrained L is written in terms of all q_j and $\dot{q}_j$, and the generalized forces of constraint

$$Q_j^C = \lambda_k\left(\frac{\partial f_k}{\partial q_j}\right). \tag{2.10}$$

The original equation (2.1) with constraint forces is therefore recovered from (2.9) and (2.10), thereby completing the circle. Simultaneous solution of (2.9) with the c-constraints (2.3) provides both the state $[q(t),\dot{q}(t)]$ of the system and the multipliers λ_k, which determine the constraint forces Q_j^C of (2.10). The $2n$ constants of integration are evaluated from the $2m = 2(n-c)$ initial values of q_j and $\dot{q}_j$ and the $2c$ constraint conditions $f_k = 0$ and $\dot{f}_k = 0$ at $t = 0$.

B. Background: Linear-velocity constraints adjoined and embedded

Linear-velocity constraints,

$$\dot{\theta}_k = g_k^{(1)}(\dot{q},q,t) = A_{kj}(q,t)\dot{q}_j + B_k(q,t) = 0 \quad (k = 1,2,\ldots,c), \tag{2.11}$$

often occur in the rotational (rolling/turning) motion of rigid bodies on a rough surface, in feedback (cruise) control and in robotics. Because (2.11) is nonintegrable, in general, the function $\theta_k(q,t)$ remains unknown. In contrast to holonomic, the nonintegrable constraints (2.11) directly restrict the kinematically possible velocities and therefore cannot be directly embedded in $L(\dot{q},q,t)$ in order to reduce the number n of generalized coordinates to m-independent coordinates. Because virtual displacements δq coincide with possible displacements dq in the limit of frozen constraints, they therefore satisfy the linear set of conditions,

$$\delta\theta_k = A_{kj}(q,t)\delta q_j = 0 \quad (k = 1,2,\ldots,c)\ , \tag{2.12}$$

which can be directly adjoined to (2.2). Thus $(L_j - \lambda_k A_{kj})\delta q_j = 0$, where all the δq_j are considered effectively as independent. Hence,

$$\boxed{L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \lambda_k\left(\frac{\partial g_k^{(1)}}{\partial \dot{q}_j}\right) \quad (j = 1,2,\ldots,n)} \tag{2.13}$$

are the standard n-equations of state for linear-velocity systems, to be solved in conjunction with the c-constraints (2.11). Because of the c restrictions (2.11) on $\dot{q}_j$, only $(n-c)$ initial values of $\dot{q}_j$ and n q_j need be assigned to provide the $2n$ constants of integration. For exactly integrable constraints, $g_k^{(E)} = \dot{f}_k(q,t)$ and the holonomic EOS (2.9) are recovered from (2.13). By taking (2.11) as c quasivelocities $\dot{\theta}_k$, the linear-velocity constraints can be embedded within a quasicoordinate $\delta\theta$-version (Sec. VII) of DLP, to give m Boltzmann–Hamel equations[14] which, in contrast to the $m + c$ equations (2.11) and (2.13), are free from Lagrange's multipliers.

Although not required for the above standard derivations of the EOS (2.9) and (2.13), the transpositional rule

$$\boxed{\delta\dot{q}_j = \frac{d}{dt}(\delta q_j) \quad \text{(Traditional commutation rule)}} \tag{2.14}$$

is traditionally assumed and used to obtain subsequent action principles[11,15] from the integrated version of (2.2). When combined with the displacement conditions (2.12), the variation $(g_k^{(1)}(\dot{q} + \delta\dot{q},q + \delta q,t) - g_k^{(1)}(\dot{q},q,t))$ to $g_k^{(1)}$ is

$$\delta g_k^{(1)} = -g_{kj}\delta q_j \tag{2.15}$$

to first order where

$$g_{kj} = \left(\frac{\partial A_{kj}}{\partial q_i} - \frac{\partial A_{ki}}{\partial q_j}\right)\dot{q}_i + \left(\frac{\partial A_{kj}}{\partial t} - \frac{\partial B_k}{\partial q_j}\right)\ . \tag{2.16}$$

The displaced states $(\dot{q} + \delta\dot{q},q + \delta q)$ for linear-velocity constraints (2.11) cannot then be possible, unless $g_{kj}\delta q_j$ vanishes which happens for integrable constraints. For exactly integrable constraints in particular, $g_k = \dot{f}_k(q,t)$ and $g_{kj} = \dot{f}_{kj} = 0$. Hamilton's principle of least action (Sec. 6A), which internally insists on a family of variationally displaced paths which are both possible and continuous, will hold[6,15] for holonomic but not for nonholonomic systems. More importantly, $\delta g_k = 0$ cannot be used to deduce displacement conditions while (2.14) is in operation. Fortunately, as we shall see, the traditional rule is not the only option for nonholonomic systems so that (2.15) need not then be the case.

C. Transpositional rule for exactly integrable systems

The explicit proof[18] of the commutation rule (2.14) considers only embedded holonomic systems where the q_i are all independent. Here we generalize it in order to include the dependent coordinates for adjoined holonomic systems. The c-constraints (2.3) can be formally used to subdivide the n coordinates into m=$(n-c)$ independent coordinates q_i with $(i=1,2,\ldots,m)$ and c dependent coordinates $\eta_k(q,t)$ with $(k=1,2,\ldots,c)$. Let $q_i(\alpha,t)$ denote the displaced configurations for the independent coordinates, where the variational parameter α is independent of t. The virtual displacements,

$$\delta q_i(t) = \left(\frac{\partial q_i(t,\alpha)}{\partial\alpha}\right)_0 \delta\alpha,$$

of the independent coordinates are taken about the actual physical trajectory labeled by $q_i(\alpha=0,t)$. Because α is independent of t, then

$$\frac{d}{dt}\left[\delta q_i(t)\right] = \frac{\partial}{\partial t}\left[\left(\frac{\partial q_i(t,\alpha)}{\partial\alpha}\right)_0 \delta\alpha\right] = \frac{\partial}{\partial\alpha}\left(\frac{\partial q_i(t,\alpha)}{\partial t}\right)_0 \delta\alpha = \delta\dot{q}_i \tag{2.17}$$

with the result[18] that (2.14) holds for all the independent δq_i. The response of the dependent displacements $\eta_k=\eta_k[q(t,\alpha),t]$ to these independent displacements is

$$\delta\eta_k(t) = \left(\frac{\partial\eta_k}{\partial q_i}\right)\left[\frac{\partial q_i(t,\alpha)}{\partial\alpha}\right]_0 \delta\alpha = \left(\frac{\partial\eta_k}{\partial q_i}\right)\delta q_i\ .$$

With the aid of the standard relations,

$$\frac{d}{dt}\left(\frac{\partial\eta_k}{\partial q_i}\right) = \frac{\partial\dot{\eta}_k}{\partial q_i}; \qquad \frac{\partial\eta_k}{\partial q_i} = \frac{\partial\dot{\eta}_k}{\partial\dot{q}_i},$$

for any function $\eta_k(q,t)$ with *independent* q_i's, and of (2.17), then

$$\frac{d}{dt}\left(\delta\eta_k\right) = \left(\frac{\partial\dot{\eta}_k}{\partial q_i}\right)\delta q_j + \left(\frac{\partial\dot{\eta}_k}{\partial\dot{q}_i}\right)\delta\dot{q}_i = \delta\dot{\eta}_k\ . \tag{2.18}$$

On relabeling η_k by q_{m+k}, the commutation relation,

$$\frac{d}{dt}(\delta q_j) = \delta\dot{q}_j \quad (j=1,2,\ldots,m,\ldots,n), \tag{2.19}$$

is proven for both the independent $(q_1,q_2,\ldots,q_m)$ and dependent $(q_{m+1},q_{m+2},\ldots,q_n)$ sets of coordinates for adjoined holonomic system. We can also show in a similar fashion that the transpositional rule,

$$\boxed{\delta\dot{f}_k - \frac{d}{dt}\left(\delta f_k\right) = \left(\frac{\partial f_k}{\partial q_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right]}, \tag{2.20}$$

holds for any function $f_k(q,t)$ with dependent and independent coordinates. The commutation rule (2.19) combined with $\delta f_k=0$, the condition (2.6) on the displacements, then implies the condition $\delta\dot{f}_k=0$ for possible states for exactly integrable constraints $g_k=\dot{f}_k=0$.

III. GENERAL KINEMATIC CONSTRAINTS: NEW RESULTS

Direct application of the d'Alembert–Lagrange principle (2.2) to systems under nonlinear kinematic constraints

$$g_k(\dot{q},q,t)=0 \quad (k=1,2,\ldots,c) \tag{3.1}$$

with a general velocity-dependence or

$$h_k(\ddot{q},\dot{q},q,t) = 0 \quad (k = 1,2,\ldots,c) \tag{3.2}$$

with general acceleration-dependence, has remained elusive in the past because the above traditional methods in Secs. II A and II B, although practical for holonomic and linear-velocity constraints, cannot be implemented. Under the commutation rule (2.14), it is impossible to extract from the nonlinear constraints (3.1) and (3.2), or from their associated variations δg_k and δh_k, the set of linear restrictions on the variational displacements δq_j needed for adjoining them, via Lagrange multipliers, to (2.2) for subsequent solution. We shall show below that a more rewarding alternative to (2.14) exists for nonholonomic systems.

From the property of possible states, the required set of linear conditions for nonholonomic systems under general velocity and acceleration constraints (3.1) and (3.2) will be established, together with a new set of transpositional relations. The EOS so obtained by adjoining these conditions to (2.2) are identical to those derived in Appendix A from the quite different Gauss principle[20] of least constraint. The reverse procedure of obtaining the EOS from transpositional relations is also viable.

A. Equations of state for homogeneous velocity constraints

We first illustrate how DLP can be directly applied to velocity constraints $g_k^{(p)}$ homogeneous to degree p in the velocities $\dot{q}_j$. For example, the velocity constraint,

$$g_k^{(2)}(\dot{q},q,t) = A_{ij}^{(k)}(q,t)\dot{q}_i\dot{q}_j = 0, \quad A_{ij}^{(k)} = A_{ji}^{(k)}\ , \tag{3.3}$$

is a homogeneous quadratic polynomial in $\dot{q}_j$. Euler's theorem on homogeneous functions with power p yields

$$\left(\frac{\partial g_k^{(p)}}{\partial \dot{q}_j}\right)\dot{q}_j = p g_k^{(p)} = 0 \quad (j = 1,2,\ldots,n) \tag{3.4}$$

from which the set of linear conditions,

$$\left(\frac{\partial g_k^{(p)}}{\partial \dot{q}_j}\right)\delta q_j = 0\ , \tag{3.5}$$

on the displacements readily arise in the linear form required for adjoining to (2.2). When (3.5) is adjoined to (2.2), the EOS are

$$\boxed{L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \lambda_k\left(\frac{\partial g_k^{(p)}}{\partial \dot{q}_j}\right) \quad (j = 1,2,\ldots,n)} \tag{3.6}$$

and the forces Q_j^C actuating the homogenous velocity constraints are $\lambda_k(\partial g_k^{(p)}/\partial \dot{q}_j)$. Using geometrical arguments and Hertz' principle of least curvature which is a geometrical version[14] of Gauss' principle of least constraint, Rund[51] has also derived (3.6). For systems under the general quadratic constraint (3.3), the EOS are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + 2\lambda_k A_{ji}^{(k)}\dot{q}_i \quad (i,j = 1,2,\ldots,n). \tag{3.7}$$

Example 1: For the nonintegrable quadratic homogeneous constraint,

$$g_1 = \dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1 = 0, \tag{3.8}$$

constructed[32] to keep the planar velocities of two particles always parallel, solution of (3.7) for no external forces predicts that $\lambda_1 = 0$, so that the forces of constraint are zero and the motion of the two particles given parallel velocities initially is free, in accord with the correct physical solution.

B. Equations of state for general velocity constraints

A desirable property in analytical dynamics is that the virtual displacements result in displaced states which are possible. Although any point on n-fold configuration space q is accessible by a geometrically possible path, the configurations accessible from a given q by dynamically possible paths lie on a manifold of only m dimensions, so that coupling to the remaining $(n-m)$ δq_j's must be obtained. We note that use of its linear-acceleration form,

$$\dot{g}_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_k}{\partial t} = 0 \quad (j = 1,2,\ldots,n), \tag{3.9}$$

automatically guarantees possible displaced states, because it leads directly to the condition,

$$\delta g_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta\dot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right)\delta q_j = \nabla_Q g_k \cdot \delta Q = 0, \tag{3.10}$$

for possible states. Because $\nabla_Q g_k$ is normal to g_k, the displacement δQ of the representative point $Q = (q,\dot{q})$ in state space is tangential to the g_k-surface and the displaced state lies on the manifold of velocity constraints g_k. Denote the m-independent and c-dependent coordinates within the $\{q_j\}$-set by q_i with $i \le m$ and by $\eta_s = q_{m+s}$, respectively, so that (3.9) decomposes as

$$\dot{g}_k = G_{ks}\ddot{\eta}_s + \left[\left(\frac{\partial g_k}{\partial \dot{q}_i}\right)\ddot{q}_i + \left(\frac{\partial g_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_k}{\partial t}\right] = 0, \tag{3.11}$$

where $G_{ks}(q,\dot{q},\eta,\dot{\eta},t) = (\partial g_k/\partial\dot{\eta}_s)$ are the elements of matrix $G = \{G_{ks}\}$, assumed to be positive definite (invertible) and where $q_j (j = 1,2,\ldots,n)$ represents all coordinates $\{q_i,\eta_s\}$. The solutions of (3.11) for the dependent accelerations are therefore

$$\ddot{\eta}_s = -\tilde{G}_{sr}\left[\left(\frac{\partial g_r}{\partial \dot{q}_i}\right)\ddot{q}_i + \left(\frac{\partial g_r}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_r}{\partial t}\right], \tag{3.12}$$

where the elements $\tilde{G}_{sr}$ of matrix $\tilde{G}$, the inverse of G, satisfy $G_{ks}\tilde{G}_{sr} = \delta_{kr}$, with $(k,r,s = 1,2,\ldots,c)$. Although the coordinate function $\eta_s = \eta_s(q_1,q_2,\ldots,q_m,t)$ is unknown for nonintegrable (3.1), the dependent displacements,

$$\delta\eta_s = \left(\frac{\partial \eta_s}{\partial q_i}\right)\delta q_i = \left(\frac{\partial \dot{\eta}_s}{\partial \dot{q}_i}\right)\delta q_i = \left(\frac{\partial \ddot{\eta}_s}{\partial \ddot{q}_i}\right)\delta q_i \quad (i = 1,2,\ldots,m), \tag{3.13}$$

can be now obtained in terms of the independent δq_i from (3.12) to give

$$\delta\eta_s = -\tilde{G}_{sr}\left(\frac{\partial g_r}{\partial \dot{q}_i}\right)\delta q_i. \tag{3.14}$$

Multiplication by G_{ks}, followed by an s-summation, yields the relation

$$\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j \equiv \left(\frac{\partial g_k}{\partial \dot{q}_i}\right)\delta q_i + \left(\frac{\partial g_k}{\partial \dot{\eta}_s}\right)\delta\eta_s = 0, \tag{3.15}$$

where η_s is now replaced by q_{m+s} and where $j = 1,2,\ldots,n$. The condition (3.10) for possible displaced states arising from (3.9) therefore yields directly the conditions (3.15) on the displacements under general velocity constraints (3.1).

On adjoining conditions (3.15) to the d'Alembert–Lagrange principle (2.2), the δq_j are regarded effectively as all free, and we obtain both the EOS,

$$\boxed{L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \lambda_k\left(\frac{\partial g_k}{\partial \dot{q}_j}\right) \quad \text{(nonholonomic EOS)}}, \tag{3.16}$$

for nonholonomic systems under general velocity constraints (3.1) and the forces $Q_j^C = \lambda_k(\partial g_k/\partial\dot{q}_j)$ of constraint. Conditions (3.15) on δq_j confirm that these ideal forces do no combined virtual work $Q_j^C\delta q_j = 0$. Both (3.15) and (3.16) cover the previous results, (2.12) and (2.13) for linear constraints and (3.5) and (3.6) for homogeneous velocity constraints. This EOS (3.16) is identical with the EOS (A16) derived in Appendix A by application of the very different Gauss principle to general velocity constraints (3.1).

Because (3.10) and (3.15) are each zero, the quantity,

$$\boxed{\delta g_k - \frac{d}{dt}\left[\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j\right] = 0 \quad (k = 1,2,\ldots,c)}, \tag{3.17}$$

is also zero and provides a new transpositional relation established in Sec. IV A.

C. Equations of state for general acceleration constraints

In a similar fashion, use of the time derivative,

$$\dot{h}_k = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\dddot{q}_j + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial h_k}{\partial t} = 0\,, \tag{3.18}$$

of the acceleration constraint (3.2), automatically guarantees possible displaced states, because it leads directly to

$$\delta h_k = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\delta\ddot{q}_j + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\delta\dot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\delta q_j = \nabla_Q h_k \cdot \delta Q = 0 \tag{3.19}$$

for possible states $Q = (q,\dot{q},\ddot{q})$. In geometrical terms, $\dot{h}_k = 0$ or its result $\delta h_k = 0$ is the tangency condition that the displaced trajectories of the representative point Q must lie on the manifold of acceleration constraints h_k. Denote, as before, the m-independent and c dependent coordinates by q_i and η_s, respectively, so that (3.18) decomposes into

$$\dot{h}_k = H_{ks}\dddot{\eta}_s + \left[\left(\frac{\partial h_k}{\partial \ddot{q}_i}\right)\dddot{q}_i + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial h_k}{\partial t}\right] = 0, \tag{3.20}$$

where $H_{ks}(q,\dot{q},\ddot{q},\eta,\dot{\eta},\ddot{\eta},t) = (\partial h_k/\partial \ddot{\eta}_s)$ are the elements of matrix $H = \{H_{ks}\}$, assumed to be positive definite (invertible). The solutions $\dddot{\eta}_s$ of (3.20) are therefore

$$\dddot{\eta}_s = -\tilde{H}_{sr}\left[\left(\frac{\partial h_r}{\partial \ddot{q}_i}\right)\dddot{q}_i + \left(\frac{\partial h_r}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial h_r}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_k}{\partial t}\right], \tag{3.21}$$

where $(i = 1,2,\ldots,m)$ and $(j = 1,2,\ldots,n)$ and where the elements $\tilde{H}_{sr}$ of matrix $\tilde{H}$, the inverse of matrix $H = \{H_{ks}\}$, satisfy $H_{ks}\tilde{H}_{sr} = \delta_{kr}$ with $(k,r,s = 1,2,\ldots,c)$. Although the coordinate function $\eta_s = \eta_s(q,t)$ is unknown for the nonintegrable (3.2), the dependent displacements,

$$\delta\eta_s = \left(\frac{\partial \eta_s}{\partial q_i}\right)\delta q_i = \left(\frac{\partial \dddot{\eta}_s}{\partial \dddot{q}_i}\right)\delta q_i, \tag{3.22}$$

may now be obtained in terms of the independent δq_i from (3.21) to give

$$\delta\eta_s = -\tilde{H}_{sr}\left(\frac{\partial h_r}{\partial \ddot{q}_i}\right)\delta q_i. \tag{3.23}$$

Multiplication by H_{ks}, followed by an s-summation, yields the relation

$$\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\delta q_j \equiv \left(\frac{\partial h_k}{\partial \ddot{q}_i}\right)\delta q_i + \left(\frac{\partial h_k}{\partial \ddot{\eta}_s}\right)\delta\eta_s = 0, \tag{3.24}$$

where η_s is now replaced by q_{m+s} and $(j = 1,2,\ldots,n)$. This is the needed set of linear conditions on the displacements to be adjoined to the d'Alembert–Lagrange principle (2.2) for nonholonomic systems under general acceleration constraints (3.2). On adjoining (3.24) to (2.2), the δq_j are then regarded effectively as all free, and we obtain both,

$$\boxed{L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \lambda_k\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right) \quad \text{(nonholonomic EOS)}}, \tag{3.25}$$

the EOS for nonholonomic systems under general acceleration constraints (3.2) and the forces of constraint $Q_j^C = \lambda_k(\partial h_k/\partial \ddot{q}_j)$. The restrictions (3.24) on δq_j ensure that the ideal forces do no combined virtual work $Q_j^C \delta q_j = 0$. With the aid of (3.9), the EOS (3.25) covers the previous result (3.16) for velocity constraints (3.1) and is identical with the EOS (A17), derived in Appendix A from the Gauss principle.

Because (3.10) and (3.15) are each zero, the quantity,

$$\boxed{\delta h_k - \frac{d^2}{dt^2}\left[\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\delta q_j\right] = 0}, \tag{3.26}$$

is also zero and provides a new transpositional relation derived in Sec. V.

IV. TRANSPOSITIONAL RELATIONS FOR VELOCITY CONSTRAINTS

The sets (3.15) and (3.24) of linear restrictions on the displacements δq_j for nonholonomic systems under general kinematic constraints (3.1) and (3.2) were derived from the basic premise that the displaced states were all possible. We will now show that the resulting relations (3.17) and (3.26), in turn, lead quite naturally to noncommuting transpositional relations for nonholonomic systems, which are quite different from the usual commutation rule (2.14) traditionally accepted in Lagrangian dynamics. We must now provide a precise meaning to $d(\delta q)/dt$. From the infinity of possible velocity sets $(\dot{q}, \dot{q}', \dot{q}'' \ldots)$ which satisfy the constraint equations (3.1) and (3.2), there exists only one set $\dot{q}$ which is realized in the actual motion determined by the EOS. Possible displacements from state $(q,\dot{q})$ in interval dt are $(dq = \dot{q}dt, d\dot{q} = \ddot{q}dt)$ and $(d'q = \dot{q}'dt, d\dot{q} = \ddot{q}dt)$. Virtual displacements from state $(q,\dot{q})$ are the differences $\delta q = (d'q - dq) = (\dot{q}' - \dot{q})dt$ and $\delta\dot{q} = (d'\dot{q} - d\dot{q}) = (\ddot{q}' - \ddot{q})dt$ of two possible displacements. Not only are all the displacement conditions, (2.6), (2.12), (3.15), and (3.24) based on frozen constraints recovered, as expected, but also the time differentials $d(\delta q)/dt = (\dot{q}' - \dot{q})$ and $d(\delta\dot{q})/dt = (\ddot{q}' - \ddot{q})$ are shown to exist.

A. Nonintegrable velocity constraints

The change in the constraint at the displaced state due to a perturbation of the physical state by a virtual displacement δq under frozen constraints is

$$\delta g_k = g_k(q + \delta q, \dot{q} + \delta\dot{q}, t) - g_k(\dot{q}, q, t),$$

which reduces, under infinitesimal displacements, to

$$\delta g_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta\dot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right)\delta q_j \quad (j = 1, 2, \ldots, n), \tag{4.1}$$

where the free and constrained velocity displacements $\delta\dot{q}_j$ are yet to be defined in terms of the δq_j. In terms of the Lagrangian derivative,

$$g_{kj} \overset{def}{\equiv} \left[\frac{d}{dt}\left(\frac{\partial g_k}{\partial \dot{q}_j}\right) - \frac{\partial g_k}{\partial q_j}\right], \tag{4.2}$$

of the constraint equation (3.1), then (4.1) provides the primary transpositional relation,

$$\boxed{\delta g_k - \frac{d}{dt}\left[\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j\right] = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right] - g_{kj}\delta q_j}, \tag{4.3}$$

derived without any assumptions. We now explore how the constrained velocity displacements $\delta\dot{q}_j$ can be obtained from δq_j in order that the property $\delta g_k = 0$ of possible displaced states can be fulfilled.

Option 1: The first option is to invoke the traditional commutation rule (2.14) for all coordinates so that (4.3) reduces to

$$\delta g_k - \frac{d}{dt}\left[\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j\right] = -g_{kj}\delta q_j \ . \tag{4.4}$$

Because (3.5) holds for linear and homogeneous velocity constraints, without requiring possible displaced states, (4.4) reduces to $\delta g_k = -g_{ki}\delta q_i$, so that the desired property of possible displaced states can never be fulfilled for nonintegrable constraints. For integrable constraints, then $g_{ki}\delta q_i = 0$ (Sec. IV A 2) and (4.4) then reduces to (2.6). When the commutation rule (2.14) is assumed for nonholonomic systems, the resulting (4.4) cannot be further utilized so that[6] "general nonholonomic constraints are completely outside the scope of even the most fundamental principle (2.2)." Also Hamilton's principle of least action will not hold[6] for nonintegrable systems. Apart from these deductions, the result (4.4) inhibits further advance, making exploration of other options necessary. Option 1 is defined by (2.14) and (4.4). For linear-velocity constraints only, (2.12) holds for the displacement $\delta\theta_k$, so that (4.4) takes the form,

$$\delta\dot{\theta}_k - \frac{d}{dt}(\delta\theta_k) = -g_{kj}\delta q_j, \quad (k = 1,2,\ldots,c) \ , \tag{4.5}$$

of a noncommuting transpositional relation in the restricted θ-space, which is equivalent to (2.14) in q-space.

Option 2: We have already shown for nonholonomic systems under velocity constraints (3.1) that the desired property $\delta g_k = 0$ of possible displaced states leads directly to the set (3.15) of linear restrictions on the δq_j, and that (3.17) holds. Consequently, the basic relation (4.3) under (3.17) reduces to the set of c transpositional relations

$$\boxed{\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right] = g_{kj}\delta q_k \quad (k = 1,2,\ldots,c)}, \tag{4.6}$$

which therefore define the velocity displacements $\delta\dot{q}_j$ in terms of the δq_j-variations in q-space. Note that (4.6) also follows from using (4.1) in (3.17). For integrable constraints, $g_{ki}\delta q_i = 0$ and (4.6) reduces to the commutation rule (2.14). In physical terms, the additional dependence of $\delta\dot{q}_j$ on the constraint g_k via $g_{ki}\delta q_i$ in (4.6) acknowledges the fact that (3.1) restricts directly the velocities of the possible states and consequently the configuration, while (2.3) for holonomic systems restricts the possible configurations and subsequently the velocities. The sum $g_{ki}\delta q_i$ will vanish only for integrable velocity constraints so that the commutation relation (2.14) is then recovered from (4.6). Option 2 is defined by (3.17) and (4.6).

(a) For linear-velocity constraints, (4.3) becomes

$$\boxed{\delta\dot{\theta}_k - \frac{d}{dt}(\delta\theta_k) = \left(\frac{\partial g_k^{(1)}}{\partial \dot{q}_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}\left(\delta q_j\right)\right] - g_k^{(1)}\delta q_j}, \tag{4.7}$$

where the Lagrangian derivative of (2.11) is (2.16). The displacement condition (2.12) is $\delta\theta_k = 0$ and the condition for possible displaced states is $\delta\dot{\theta}_k = 0$ so that the commutation rule,

$$\delta\dot{\theta}_k - \frac{d}{dt}(\delta\theta_k) = 0, \quad (k = 1,2,\ldots,c), \tag{4.8}$$

in the restricted θ-space is equivalent to the noncommuting relation (4.6) in q-space, which may also be verified by the following calculation. Solve (2.11) for the c $(= n - m)$ dependent velocities $\dot{\eta}_k$ in terms of the m-independent velocities $\dot{q}_i$ and all the n independent and dependent coordinates (q_i, η_k) to give

$$\dot{\theta}_k = G_k(q,\eta,\dot{q},\dot{\eta},t) = C_{ki}(q,\eta,t)\dot{q}_i + D_k(q,\eta,t) - \dot{\eta}_k = 0, \quad (i = 1,2,\ldots,m) \ , \tag{4.9}$$

$$\delta\theta_k = C_{ki}(q,\eta,t)\delta q_i - \delta\eta_k = 0. \tag{4.10}$$

Direct calculation of $\delta\dot{\eta}_k - d(\delta\eta_k)/dt$ leads to a cumbersome result which may be written concisely as,

$$\left(\frac{\partial G_k}{\partial \dot{q}_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right] = G_{kj}\delta q_j, \quad (j = 1,2,\ldots,n), \tag{4.11}$$

in agreement with (4.6), where G_{kj} is the Lagrangian derivative of (4.9).

(b) Application of (3.5) and (4.1) to the nonintegrable homogeneous constraint (3.8) yields

$$\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j = \dot{y}_2\delta x_1 + \dot{x}_1\delta y_2 - \dot{y}_1\delta x_2 - \dot{x}_2\delta y_1 = 0\ ,$$

$$\delta g_1 = \dot{y}_2\delta\dot{x}_1 + \dot{x}_1\delta\dot{y}_2 - \dot{y}_1\delta\dot{x}_2 - \dot{x}_2\delta\dot{y}_1\ .$$

On setting $\delta g_1 = 0$, results in agreement with those calculated from the (4.6) are also obtained.

The two options above are further examined and illustrated in Appendix B for the case of the nonholonomic penny rolling and turning on an inclined plane.

1. Subrules

Because (4.6) is a set of only $c = (n - m)$ equations for the $n = (m + c)$ unknown $\delta\dot{q}_j$, we are therefore at liberty to specify that the commutation relation (2.14) is obeyed by the m-independent velocity displacements $\delta\dot{q}_r$. Then (4.6) is reduced to the set of c equations,

$$G_{ks}\left[\delta\dot{q}_s - \frac{d}{dt}(\delta q_s)\right] = g_{kj}\delta q_j, \quad G_{ks} = \left(\frac{\partial g_k}{\partial \dot{q}_s}\right), \quad (s = m+1, m+2, \ldots, n), \tag{4.12}$$

for the c dependent velocity displacements. The solution of (4.12) is

$$\begin{aligned} &\delta\dot{q}_r - \frac{d}{dt}(\delta q_r) = 0, \quad (r = 1,2,\ldots,m),\\ &\delta\dot{q}_s - \frac{d}{dt}(\delta q_s) = \tilde{G}_{sk}g_{kj}\delta q_j\ , \quad (s = m+1, m+2, \ldots, n), \end{aligned} \tag{4.13}$$

where the elements $\tilde{G}_{sk}$ of the $(c \times c)$ inverse matrix $\tilde{G}$ satisfy $\tilde{G}_{sk}G_{kj} = \delta_{sj}$. The subrules (4.13), based on (4.6), show how to evaluate the independent and dependent velocity displacements from δq_j.

2. Integrable velocity constraints

When the linear-velocity constraint (2.11) is exactly integrable then,

$$g_k^{(E)}(q,\dot{q},t) = \dot{f}_k(q,t) = \left(\frac{\partial f_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial f_k}{\partial t}, \tag{4.14}$$

integrates (without the need of an integrating factor) to holonomic form (2.3). With the aid of the identities,

$$\frac{\partial \dot{f}_k}{\partial \dot{q}_j} = \frac{\partial f_k}{\partial q_j}, \tag{4.15}$$

$$\frac{\partial \dot{f}_k}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial f_k}{\partial q_j}\right), \tag{4.16}$$

easily proven from (4.14) on treating all the q_j's as independent, the Lagrangian derivatives,

$$g_{kj}^{(E)} = \dot{f}_{kj} = \left[\frac{d}{dt}\left(\frac{\partial \dot{f}_k}{\partial \dot{q}_j}\right) - \frac{\partial \dot{f}_k}{\partial q_j}\right] = \left[\frac{d}{dt}\left(\frac{\partial f_k}{\partial q_j}\right) - \frac{\partial \dot{f}_k}{\partial q_j}\right] = 0, \tag{4.17}$$

of (4.14) always vanish. Otherwise, the explicit Lagrangian derivative (2.16) vanishes when

$$\frac{\partial A_{kj}}{\partial q_i} = \frac{\partial A_{ki}}{\partial q_j}\ ; \quad \frac{\partial A_{kj}}{\partial t} = \frac{\partial B_k}{\partial q_j} \quad (i, j = 1,2,\ldots,n), \tag{4.18}$$

which are the precise conditions deduced[6,16,17] from (4.14), that the linear constraint (2.11) is *exactly* integrable (without the need of an integrating factor) to the holonomic form (2.3). The primary relation (4.3) then covers the transpositional rule (2.20) for $f_k(q,t)$. As an aside, note that the zero Lagrangian derivative (4.17) is the essential reason why the extended Lagrangian $\tilde{L} = L + \dot{f}_k$ also satisfies Lagranges' equations because then $\tilde{L}_j = L_j = Q_j^{NP}$. Sometimes the linear-velocity constraint (2.11) with $B_k = 0$ can be rendered in exact form via the integrating factor,

$$\Phi_k(q_i) = \exp\left[\int \left(\frac{\partial A_{ki}}{\partial q_j} - \frac{\partial A_{kj}}{\partial q_i}\right)\frac{dq_i}{A_{kj}}\right], \tag{4.19}$$

provided the integrand is a function only of a specific coordinate q_i. For constraints requiring an integrating factor for integration, the sum $g_{kj}\delta q_j$ in (4.6) will vanish. The Lagrangian derivatives of constraints written exactly as $g_k^{(E)} = \Phi_k g_k = \dot{f}_k(q,t)$ separately vanish.

Example 2: For example, consider the (scleronomic) linear-velocity constraint

$$g_1 = (4q_1 + 3q_2^2)\dot{q}_1 + (2q_1q_2)\dot{q}_2 = 0 \tag{4.20}$$

so that the displacements are related by

$$(4q_1 + 3q_2^2)\delta q_1 + (2q_1q_2)\delta q_2 = 0\ . \tag{4.21}$$

With the use of $\delta g_1 = 0$ for possible states, direct calculation yields

$$(4q_1 + 3q_2^2)\left[\delta\dot{q}_1 - \frac{d}{dt}(\delta q_1)\right] + (2q_1q_2)\left[\delta\dot{q}_2 - \frac{d}{dt}(\delta q_2)\right] = 4q_2\left(\dot{q}_2\delta q_1 - \dot{q}_1\delta q_2\right), \tag{4.22}$$

which is identical with that obtained from the general relation (4.6). With the aid of (4.20) and (4.21), the RHS of (4.22) reduces to zero. The dependent and independent velocity displacements then obey the commutation relations (2.19). The underlying reason is that (4.20) is integrable via the integrating factor $\Phi_1 = q_1^2$ obtained from (4.19). The exact form of (4.20) is then

$$\dot{\theta}_1 = g_1^{(E)} = (4q_1^3 + 3q_1^2q_2^2)\dot{q}_1 + (2q_1^3q_2)\dot{q}_2 = 0, \tag{4.23}$$

which integrates exactly to $\theta_1(q_1,q_2) = q_1^2(q_1^2 + q_1q_2^2) + c_1(\text{constant})$. For exact constraints (4.23), the individual Lagrange derivatives $g_{kj}^{(E)}$ separately vanish so that (4.6) reduces to the commutation relation (2.19) for holonomic constraints. For integrable constraints (4.20), the sum $g_{kj}\delta q_j$, the RHS of (4.22), vanishes.

Example 3: The (rheonomic) linear-velocity constraint

$$g_2 = (q_2\dot{q}_1 + q_1\dot{q}_2)C(t) - q_1q_2\left(\frac{dC}{dt}\right) = 0, \tag{4.24}$$

where $C(t)$ is any general function of t, is integrable, as confirmed by the satisfaction of $g_{2j}\delta q_j = 0$. The constraint $g_2^{(E)} = q_1^{-2}q_2^{-2}g_2$ is exactly integrable, as confirmed by $g_{2j}^{(E)} = 0$ or by the explicit conditions (4.18).

For integrable constraints, the general relations (4.13) reduce to the commutation rule (2.14).

B. Transpositional form of the d'Alembert–Lagrange principle

On replacing g_k by L in (4.3), the d'Alembert–Lagrange principle (2.2) can be expressed in its basic transpositional form as

$$L_j\delta q_j = \frac{d}{dt}\left[\left(\frac{\partial L}{\partial\dot{q}_j}\right)\delta q_j\right] - \delta L + \left(\frac{\partial L}{\partial\dot{q}_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}\left(\delta q_j\right)\right] = Q_j^{NP}\delta q_j, \tag{4.25}$$

where the displacement δq changes the Lagrangian L by

$$\begin{aligned}\delta L &= L(q+\delta q,\dot{q}+\delta\dot{q},t) - L(q,\dot{q},t),\\ &= \left(\frac{\partial L}{\partial\dot{q}_j}\right)\delta\dot{q}_j + \left(\frac{\partial L}{\partial q_j}\right)\delta q_j\ ,\end{aligned} \tag{4.26}$$

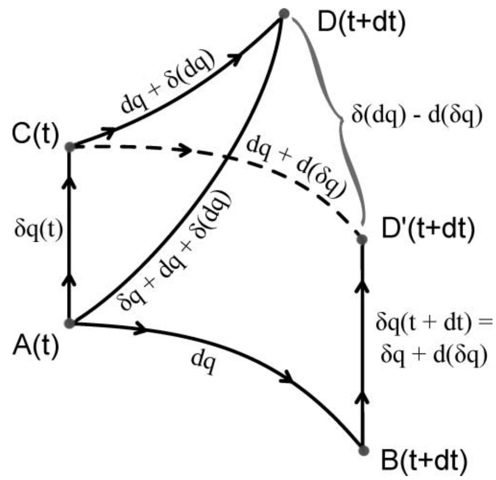

FIG. 1. Physical states $A(t)$ and $B(t+dt)$ are perturbed to C and D' under nonintegrable constraints. (a) Commutation rule (2.14) implies a continuous path CD of nonpossible displaced states C and D. (b) Noncommutation rule (4.6) implies two possible displaced states C and D', but CD' is discontinuous. The vector from D' to D is $\delta(dq) - d(\delta q)$ in the nq-space. For integrable constraints, $D \to D'$ and $DCABD'$ closes at D' to give continuous possible displaced states.

which will depend on transpositional rule adopted.

C. The $\delta(dq)$–$d(\delta q)$-relation

Although it is not required for the derivation of the EOS for holonomic and linear-velocity systems, the commutation rule (2.14) is traditionally assumed in the further development of constrained principles from (4.25), on the basis that "the variations are contemporaneous,"[15] so that the d and δ operations are assumed to commute for all kinds of constraints. In a strict sense, however, contemporaneous means only that $d(\delta t) = \delta(dt) = 0$, and not that $d(\delta q_j) = \delta(dq_j)$ necessarily holds true in constrained velocity space. The difference between the rules (2.14) and (4.6) for calculation of velocity displacements can be illustrated geometrically in n dimensional q-space, in a manner similar to Greenwood.[16] In Fig. 1, AB is a real displacement $dq = \dot{q}dt$ along the actual physical continuous path during time interval $(t, t+dt)$. A virtual displacement $\delta q(t)$ to state $A(q,\dot{q})$ produces state $C(q+\delta q, \dot{q}+\delta\dot{q})$. After interval dt, this state C evolves with initial velocity $(\dot{q}+\delta\dot{q})$ to state D at time $t+dt$, along the varied path of length $CD = (\dot{q}+\delta\dot{q})dt = dq + \delta(dq)$. A virtual displacement $\delta q(t+dt) = \delta q + d(\delta q)$ at B perturbs state $B(q(t+dt), \dot{q}(t+dt))$ to a state D' which is *different* from D. The difference in the configurations $[q+\delta q+dq+\delta(dq)]$ and $[q+dq+\delta q+d(\delta q)]$ of states D and D', which respectively originate from virtual displacements acting at A and B, is the length $DD' = \delta(dq) - d(\delta q)$. This difference varies with time at the rate of $(DD')/dt$, which is therefore $\delta\dot{q} - d(\delta q)/dt$, as determined by (4.13). By considering accelerations, it can be shown, in a similar way, that the difference in the velocities at D and D' is $\delta(d\dot{q}) - d(\delta\dot{q})$, which increases at the rate of $\delta\ddot{q} - d(\delta\dot{q})/dt$, as determined by the transpositional relation (4.31) to be derived in Sec. IV D.

The commutation rule (2.14) for nonintegrable systems therefore implies coincident D and D' so that the quadrilateral $DCABD'$ closes at $D = D'$ for all time. The $\delta q(t)$ displacement at A therefore generates a single continuous displaced path CD which is not possible, because the nonintegrable constraints are not satisfied along CD. The different transpositional rule (4.6) implies that the quadrilateral remains open at different end-points D and D' which vary with t. Although possible displaced states exist at C and at D', CD' is not a continuous path. The two rules (2.14) and (4.6) therefore imply that the paths are either continuous and impossible or discontinuous and possible, respectively, but not both, with the result that integral variational principles (as Hamilton's) which rely on continuous possible paths do not then pertain. Under the subrules (4.13) consistent with (4.6), the quadrilateral, although closed in m-dimensional q-space of the independent coordinates, remains open in the $(n-m)$-dimensional q-space of the dependent coordinates. It is only for integrable velocity constraints that the quadrilateral becomes closed over the full n-dimensional q-space when $D \to D'$ so that CD' now becomes a continuous possible path.

D. Higher-order transpositional relations

The linear acceleration function,

$$\dot{g}_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_k}{\partial t}\ , \tag{4.27}$$

integrates exactly to the general velocity constraint (3.1). Construct $\delta\dot{g}_k$ from (4.27) and $d(\delta g_k)dt$ from either (4.1) or (4.3). With the aid of the identities

$$\left(\frac{\partial \dot{g}_k}{\partial \ddot{q}_j}\right) = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right), \tag{4.28}$$

$$\left(\frac{\partial \dot{g}_k}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial g_k}{\partial \dot{q}_j}\right) + \frac{\partial g_k}{\partial q_j}, \tag{4.29}$$

$$\left(\frac{\partial \dot{g}_k}{\partial q_j}\right) = \frac{d}{dt}\left(\frac{\partial g_k}{\partial q_j}\right), \tag{4.30}$$

proven easily from (4.27), the transpositional relation,

$$\boxed{\delta\dot{g}_k - \frac{d}{dt}(\delta g_k) = \left(\frac{\partial g_k}{\partial q_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right] + \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta\ddot{q}_j - \frac{d}{dt}(\delta\dot{q}_j)\right]}, \tag{4.31}$$

is then obtained for kinematic constraints (3.1), and is analogous to (2.20) for holonomic systems. With the aid of the primary relation (4.3), (4.31) may be alternatively expressed as

$$\boxed{\delta\dot{g}_k - \frac{d^2}{dt^2}\left[\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j\right] = \left(\frac{\partial \dot{g}_k}{\partial \dot{q}_j}\right)\left(\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right) + \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta\ddot{q}_j - \frac{d^2}{dt^2}(\delta q_j)\right] - \frac{d}{dt}(g_{kj}\delta q_j)}, \tag{4.32}$$

which is a higher-order version of (4.3). For exactly integrable constraints, $g_k = \dot{f}_k$. Then both (4.31) and (4.32) reduce to

$$\boxed{\delta\ddot{f}_k - \frac{d}{dt}(\delta\dot{f}_k) = \left(\frac{\partial \dot{f}_k}{\partial q_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right] + \left(\frac{\partial f_k}{\partial q_j}\right)\left[\delta\ddot{q}_j - \frac{d}{dt}(\delta\dot{q}_j)\right]}, \tag{4.33}$$

$$\boxed{\delta\ddot{f}_k - \frac{d^2}{dt^2}(\delta f_k) = 2\left(\frac{\partial \dot{f}_k}{\partial q_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right] + \left(\frac{\partial f_k}{\partial q_j}\right)\left[\delta\ddot{q}_j - \frac{d^2}{dt^2}(\delta q_j)\right]}, \tag{4.34}$$

which are the higher-order versions of (2.20). Two distinct families, (A) with members (4.3), (4.32), and (4.34) and (B) with members (2.20), (4.31), and (4.33), of transpositional relations have now been established. When (4.27) is set to zero and used instead of (3.1), then possible states ($\delta g_k = 0$) are automatically implied.

V. TRANSPOSITIONAL RELATIONS FOR ACCELERATION CONSTRAINTS

A. Primary transpositional rule

The change in the acceleration constraints,

$$h_k(\ddot{q},\dot{q},q,t) = 0 \quad (k = 1,2,\ldots,c), \tag{5.1}$$

due to the δq-displacement is

$$\delta h_k = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\delta \ddot{q}_j + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\delta \dot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\delta q_j \ . \tag{5.2}$$

Denote the first- and second-time derivatives by $(\ldots)'$ and $(\ldots)''$, respectively. Then

$$\delta h_k - \left(\frac{\partial h_k}{\partial \ddot{q}_j}\delta q_j\right)'' = \delta h_k - \left[\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)(\delta q_j)'' + 2\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)'(\delta q_j)' + \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)''\delta q_j\right] ,$$

which with the aid of (5.2), produces the primary transpositional relation

$$\boxed{\delta h_k - \left(\frac{\partial h_k}{\partial \ddot{q}_j}\delta q_j\right)'' = \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\left[\delta \dot{q}_j - (\delta q_j)'\right] + \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\left[\delta \ddot{q}_j - (\delta q_j)''\right] - \Delta h_k} \tag{5.3}$$

for acceleration constraints, where the end term is

$$\boxed{\Delta h_k = \left[2\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)' - \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\right](\delta q_j)' + \left[\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)'' - \left(\frac{\partial h_k}{\partial q_j}\right)\right]\delta q_j}. \tag{5.4}$$

The physical meaning of (5.3) is made apparent for exact constraints $h_k = \dot{g}_k$. With the aid of the identities (4.28)–(4.30), Δh_k simply reduces to $(g_{kj}\delta q_j)'$ and (5.3) then reproduces the higher-order transpositional relation (4.32) previously derived for velocity constraints. We can also show that (5.3) with $h_k = \dot{g}_k$ minus the time derivative of (4.3) reproduces (4.31). When the acceleration constraints are exactly integrable to holonomic constraints, then

$$h_k = \ddot{f}_k = \left(\frac{\partial f_k}{\partial q_j}\right)\ddot{q}_j + \left(\frac{\partial^2 f_k}{\partial q_i \partial q_j}\right)\dot{q}_i\dot{q}_j + 2\left(\frac{\partial^2 f_k}{\partial t \partial q_j}\right)\dot{q}_j + \frac{\partial^2 f_k}{\partial t^2}$$

so that $\Delta h_k = 0$, via use of the identities

$$\frac{\partial \ddot{f}_k}{\partial \ddot{q}_j} = \frac{\partial f_k}{\partial q_j} \ ; \quad \frac{\partial \ddot{f}_k}{\partial \dot{q}_j} = 2\frac{\partial \dot{f}_k}{\partial q_j},$$

$$\left(\frac{\partial \ddot{f}_k}{\partial \ddot{q}_j}\right)' = \frac{\partial \dot{f}_k}{\partial q_j} \ ; \quad \left(\frac{\partial \ddot{f}_k}{\partial \ddot{q}_j}\right)'' = \frac{\partial \ddot{f}_k}{\partial q_j} \ .$$

For acceleration constraints exactly integrable to holonomic form (5.3) reduces to the previous rule (4.34). Relation (5.3) adds another member to the A-family, (4.3), (4.32), and (4.34), of transpositional relations.

1. Subrules

In Sec. III C, it was shown that (3.26) holds for acceleration constraints. As a result, the primary rule (5.3) then reduces to the transpositional relation

$$\left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\left[\delta \dot{q}_j - (\delta q_j)'\right] + \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\left[\delta \ddot{q}_j - (\delta q_j)''\right] = \Delta h_k \quad (k = 1,2,\ldots,c). \tag{5.5}$$

This set of c equations (5.5) can only be solved for the c dependent acceleration displacements $\delta \ddot{q}_j$ provided the commutation relations,

$$\begin{aligned} \delta \ddot{q}_j &= \frac{d}{dt}(\delta \dot{q}_j) \quad (j = 1,2,\ldots,m), \\ \delta \dot{q}_j &= \frac{d}{dt}(\delta q_j) \quad (j = 1,2,\ldots,n), \end{aligned} \tag{5.6}$$

are obeyed by the m-independent acceleration displacements and all n velocity displacements. The dependent acceleration displacements then satisfy

$$\delta \ddot{q}_j - \left(\delta q_j\right)'' = \tilde{H}_{jk}\Delta h_k; \quad H_{ki} = \left(\frac{\partial h_k}{\partial \ddot{q}_i}\right) \quad (j = m+1, m+2, \ldots, n), \tag{5.7}$$

where $\tilde{H}_{jk}H_{ki} = \delta_{ji}$ is satisfied by elements of the $(c \times c)$ matrix $\tilde{H}$, the inverse of H.

B. Higher-order transpositional relations

On constructing $\delta \dot{h}_k$ from (3.2) and $(\delta h_k)'$ from (5.2), then, with aid of the identities,

$$\left(\frac{\partial \dot{h}_k}{\partial \dddot{q}_j}\right) = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right) \; ; \quad \left(\frac{\partial \dot{h}_k}{\partial \ddot{q}_j}\right) = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)' + \frac{\partial h_k}{\partial \dot{q}_j} \, ,$$

$$\left(\frac{\partial \dot{h}_k}{\partial q_j}\right) = \left(\frac{\partial h_k}{\partial q_j}\right)' \; ; \quad \left(\frac{\partial \dot{h}_k}{\partial \dot{q}_j}\right) = \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)' + \frac{\partial h_k}{\partial q_j} \, ,$$

proven from (5.1), the higher-order transpositional rule,

$$\boxed{\delta \dot{h}_k - (\delta h_k)' = \left(\frac{\partial h_k}{\partial q_j}\right)\left[\delta \dot{q}_j - (\delta q_j)'\right] + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\left[\delta \ddot{q}_j - (\delta \dot{q}_j)'\right] + \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\left[\delta \dddot{q}_j - (\delta \ddot{q}_j)'\right]}, \tag{5.8}$$

is obtained for acceleration constraints (3.2). When the acceleration constraints are given by (4.27), they are linear and exactly integrable to general velocity constraints g_k. Application of (5.8) provides the relation,

$$\boxed{\delta \ddot{g}_k - (\delta \dot{g}_k)' = \left(\frac{\partial g_k}{\partial q_j}\right)'\left[\delta \dot{q}_j - (\delta q_j)'\right] + \left[\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)' + \frac{\partial g_k}{\partial q_j}\right]\left[\delta \ddot{q}_j - (\delta \dot{q}_j)'\right] + \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta \dddot{q}_j - (\delta \ddot{q}_j)'\right]}, \tag{5.9}$$

for $\delta \ddot{g}_k$, the fifth member of the B-family of transpositional relations, (2.20), (4.31), (4.33), and (5.8). All of the higher-order transpositional relations in the (A, B) hierarchies reduce, in the appropriate limit, to the ones below them.

VI. CONSTRAINED PRINCIPLES

A. Constrained Hamilton's Principle

Hamilton's principle is valid provided the chosen varied paths $q(t) + \delta q(t)$ are continuous and possible, being compatible with the constraints by satisfying $\delta g_k = g_k(q + \delta q, \dot{q} + \delta \dot{q}, t) = 0$ at all times. Because the traditional commutation relation (2.19) implies $\delta g_k \neq 0$, the principle does not hold[6,15–17] for nonintegrable constraints. In the light that the present transpositional rule (4.6) for nonholonomic systems accommodates possible states, let us therefore re-examine the validity of Hamilton's principle for general velocity constraints (3.1).

Integrate (4.25) over time and assume, as is customary for integral principles, that the displacements (surface terms) vanish at *both* fixed end points $t_{1,2}$, then

$$\int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}_j}\right)\left[\delta \dot{q}_j - \frac{d}{dt}(\delta q_j)\right] dt - \int_{t_1}^{t_2} Q_j^{NP} \delta q_j dt \tag{6.1}$$

in general. Under the sub-rules (4.13) for velocity constraints (3.1), the integrated version of (4.25) provides,

$$\int_{t_1}^{t_2} \left[\delta L - \left(\frac{\partial L}{\partial \dot{q}_s}\right) \tilde{G}_{sk}(g_{kj} \delta q_j) + Q_j^{NP} \delta q_j\right] dt = 0, \quad (s = m+1, m+2, \ldots, n), (j = 1, 2, \ldots, n), \tag{6.2}$$

which is the present version of Hamilton's integrated principle under the transpositional rule (4.13). As a test, adjoin the constraint conditions $\delta g_k = 0$ under the present transpositional rule (4.6) by subtracting,

$$\mu_k(t) \delta g_k = \mu_k \left(\frac{\partial g_k}{\partial \dot{q}_j} \delta q_j\right)' = \left(\mu_k \frac{\partial g_k}{\partial \dot{q}_j} \delta q_j\right)' - \dot{\mu}_k \left(\frac{\partial g_k}{\partial \dot{q}_j}\right) \delta q_j = 0, \tag{6.3}$$

from the integrand of (6.2). Then,

$$\int_{t_1}^{t_2} \left[L_j - \dot{\mu}_k \left(\frac{\partial g_k}{\partial \dot{q}_j}\right) - Q_j^{NP}\right] \delta q_j(t) dt = 0, \tag{6.4}$$

where δq_j can be regarded as independent. The correct EOS (3.16) then emerges from (6.4) and (6.2) is verified.

The traditional form of Hamilton's integrated principle,

$$\int_{t_1}^{t_2} \delta L \, dt = 0 \quad \text{(Hamilton's integrated principle)}, \tag{6.5}$$

now follows from (6.2) on assuming integrable constraints when $g_{kj}\delta q_j = 0$ and workless non-potential forces when $Q_j^{NP}\delta q_j = 0$. The virtual-displacement operator δ commutes with time t and (6.5) then becomes a real variational principle,

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt = 0 \quad \text{(Hamilton's principle)}, \tag{6.6}$$

based on finding stationary values of the action S. Constraints are acknowledged either by imposing on (6.5) the side conditions (6.3) on the displacements δq or by imposing on (6.6) the actual constraint equations $g_k = 0$ as side conditions. The varied states $q(t,\alpha), \dot{q}(t,\alpha)$ selected in (6.6) must then all be possible for all variational parameters α. Non-integrable constraints are automatically excluded in (6.6) because the transpositional relation (4.13) prevents passage from (6.2) to (6.5). A basic theorem in the calculus of variations states that the extremum to the functional $S = \int L \, dt$ subject to auxiliary conditions (3.1) for possible displaced paths can be determined by,

$$\delta \tilde{S} = \delta \int_{t_1}^{t_2} (L - \mu_k g_k^{(I)}) \, dt = 0, \quad \text{(Hamilton's constrained principle)}, \tag{6.7}$$

which is the free variation of the constrained action $\tilde{S}$ without any side conditions imposed. With the aid of (6.3) and $g_{kj}\delta q_j = 0$ for integrable constraints, the correct EOS, $L_j = \dot{\mu}_k(\partial g_k^{(I)})/\partial \dot{q}_j)$, is recovered from (6.7). Note that it is not necessary to write $g_k^{(I)}$ in exact form to maintain validity of (6.7) for integrable constraints.

When the commutation relation (2.14) is used in (6.1), then the integral principle (6.5) is satisfied[6,15] even for non-integrable constraints g_k. The subsequent advance from (6.5) for non-integrable constraints to the variational principle (6.7) is however prevented by the fact that the side conditions $g_k = 0$ cannot be realized for the varied states because the commutation relation implies that the variational path is composed of non-possible states with $\delta g_k \neq 0$. In contrast to (2.14), the transpositional relation (4.13) implies that (6.5) is valid only for integrable constraints which, in turn, imply possible states which provide the continuous variational path required for the validity of (6.7). Under either rule, (2.14) or (4.13) however, the conclusion remains the same in that the physical state of a nonholonomic system does not result from a stationary value of the constrained action.

B. Axiomatic constrained principles

To illustrate directly the failure of Hamilton's constrained principle (6.7) for non-integrable constraints, simply replace the integrable constraint $g_k^{(I)}$ in (6.7) by the nonintegrable constraint g_k.

(a) Then (6.7), with the aid of the commutation rule (2.14) in (6.1) and the condition

$$\mu_k(t)\delta g_k = \left(\mu_k \frac{\partial g_k}{\partial \dot{q}_j}\delta q_j\right)' - [\mu_k g_k]_j \delta q_j \neq 0, \tag{6.8}$$

yields $L_j = (\mu_k g_k)_j$, the Lagrangian derivative of $\mu_k g_k$ which gives,

$$L_j = \dot{\mu}_k \left(\frac{\partial g_k}{\partial \dot{q}_j}\right) + \mu_k \, g_{kj} \ , \tag{6.9}$$

the n-equations of state to be solved in conjunction with (3.1). The end-term $(\mu_k g_{kj})$ however prevents agreement with the correct result (3.16). Equations (6.9), first proposed in 1966 by Ray[39] and then retracted,[39] were later re-discovered[40,41] in 1983 and termed the vakonomic equations. Because of the appearance of both $\dot{\mu}_k$ and μ_k in (6.9), knowledge of the $2n$ initial values of $(q, \dot{q})$, must somehow be supplemented by specifying the c-Lagrangian multipliers, μ_k at $t = 0$ i.e., the forces of constraint at $t = 0$ must be known in order to determine full solution of (6.9). Under an appropriate choice of these initial conditions, it may be possible to reproduce correct results for a particular system. It has recently been shown[4] that the vakonomic solutions, although coinciding in some cases for certain initial data, differs, in most cases, from the DLP dynamics of nonholonomic systems. Arnold *et al.*[41] have also remarked on the conflicting (paradoxical) solutions obtained for the "vakonomic" and "nonholonomic" ice-skaters on an inclined plane. More doubt has been raised[42] on the overall effectiveness of vakonomic mechanics for velocity constraints. In general, (6.9) fails to reproduce the traditional formula (2.13) for linear-velocity constraints (2.11) for which the conditions (2.12) were already known. Comparison of (6.9) with the correct result (3.16) reveals that axiom will, in general be valid *only* for integrable velocity constraints when $g_{kj}\delta q_j$ intrinsic to calculation of (6.7) disappears.

(b) Under the present transpositional rule (4.6), (6.1) yields

$$\int_{t_1}^{t_2} \delta L \, dt = -\int_{t_1}^{t_2} L_j \, \delta q_j dt + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}_s}\right) \tilde{G}_{sk}(g_{kj}\delta q_j) dt, \tag{6.10}$$

so that (6.7) with (6.3) provides the equations of state,

$$L_j = \dot{\mu}_k \left(\frac{\partial g_k}{\partial \dot{q}_j}\right) + \left(\frac{\partial L}{\partial \dot{q}_s}\right) \tilde{G}_{sk} \; g_{kj} \ , \tag{6.11}$$

where, in contrast to (6.9) only $\dot{\mu}_k$ now enters. However, the end-term still prevents agreement with the correct result (3.16). This is because (6.3) should be adjoined to the integrand of (6.2) rather than to δL of (6.10). Hamilton's constrained principle for general g_k does not work unless the constraints are integrable when $g_{kj}\delta q_j = 0$, as already proven in Sect. VI A.

VII. TRANSPOSITIONAL RELATIONS FOR QUASIVELOCITIES IN LINEAR-VELOCITY CONSTRAINTS

The present work on transpositional relations is geared towards fulfilling the property of possible states for general nonholonomic systems. Previous work[16,22,38,39] dealt with transpositional relations between true and quasivelocities in order to provide an alternative derivation of the Boltzmann–Hamel equation,[14] in which the linear-velocity constraints are embedded and therefore free of Lagrange multipliers. Instead of the m-independent velocities $\dot{q}_j$, the m nonzero quasivelocities,

$$\dot{\theta}_i = A_{ij}(q,t)\dot{q}_j + B_i(q,t) \quad (i, j = 1,2,\ldots,n) \ , \tag{7.1}$$

can be adopted as independent velocities, together with c linear-velocity constraints,

$$\dot{\theta}_s = A_{sj}(q,t)\dot{q}_j + B_s(q,t) = 0, \quad (s = m+1, m+2, \ldots, n), \tag{7.2}$$

which can be taken to be the remaining c dependent quasivelocities which are now zero. Because (7.1) are linear in $\dot{q}$ and, in general, are nonintegrable, $\theta_i = \theta_i(q,t)$ are unknown coordinate functions and therefore unusable as true generalized coordinates. But θ_i have known nonzero displacements

$$\delta\theta_i = \left(\frac{\partial \theta_i}{\partial q_j}\right)\delta q_j = \left(\frac{\partial \dot{\theta}_i}{\partial \dot{q}_j}\right)\delta q_j = A_{ij}(q,t)\delta q_j \ . \tag{7.3}$$

Solutions of (7.1)–(7.3) are

$$\dot{q}_j = \tilde{A}_{ji}(q,t)\left[\dot{\theta}_i - B_i(q,t)\right], \tag{7.4}$$

$$\delta q_j = \tilde{A}_{ji}(q,t)\delta\theta_i \quad (i, j = 1,2,\ldots,n), \tag{7.5}$$

where $\tilde{A}_{jr}A_{ri} = \delta_{ji}$ is satisfied by elements of the $(n \times n)$ inverse matrix $\tilde{A}$. The transpositional relation (4.7) for the c-constraints (2.11) can then be extended to cover the independent quasivelocities in (7.1) to give the primary rule

$$A_{ij}\left[\delta\dot{q}_j - \frac{d}{dt}\left(\delta q_j\right)\right] - \left[\delta\dot{\theta}_i - \frac{d}{dt}(\delta\theta_i)\right] = \dot{\theta}_{ij}\tilde{A}_{jr}\delta\theta_r \quad (i, j, r = 1,2,\ldots,n) \ , \tag{7.6}$$

where $\dot{\theta}_{ij}$ represent the Lagrangian derivatives (2.16) with respect to $(q,\dot{q})$ of the m-independent quasivelocities (7.1) and the c-constraints (2.11). Relation (7.6) agrees with that previously derived[16,49,50] in terms of Hamel coefficients (Sec. VI B). In previous work[16,49], it was then necessary, for further advance, to choose either of the q- or θ-commutation rules for all coordinates in (7.6). The present work is based on possible displaced states so that the independent and dependent velocity-displacements $\delta\dot{q}_j$ are governed by additional transpositional relations (4.13) which can be used in (7.6) to provide the further relations,

$$\begin{aligned}
&\delta\dot{\theta}_r - \frac{d}{dt}(\delta\theta_r) = -\dot{\theta}_{ri}\tilde{A}_{ik}\delta\theta_k, \quad (i,k,r = 1,2,\ldots,m),\\
&\delta\dot{\theta}_s - \frac{d}{dt}(\delta\theta_s) = 0, \quad (s = m+1,m+2,\ldots,n),
\end{aligned} \tag{7.7}$$

for the quasi-velocity displacements $\delta\dot{\theta}_j$. It is worth noting that the present work partitions the independent and dependent velocity displacements and that commutation for the independent $\delta\dot{q}_r$ implies non-commutation for the independent $\delta\dot{\theta}_r$, and vice-versa for the corresponding dependent velocity displacements. The combination of (4.13) and (7.7) also reveals that it is not necessary to choose[16,49] either of the q- or θ-commutation rules for all coordinates. Rather, the q-commutation rule now holds for independent velocities while the θ-commutation rule holds for the dependent quasi-velocities, thereby guaranteeing all possible states. Because of the new transpositional relations (4.13), generalization of the previous work[16,49,50] to cover possible states is now possible.

A. Transpositional form of DLP in quasivelocities

The Lagrangian is expressed in terms of the state set $(q,\dot{\theta})$ as

$$L\left[q,\dot{q}(\dot{\theta},q,t),t\right] = \tilde{L}(q,\dot{\theta}(\dot{q},q,t),t)\ . \tag{7.8}$$

With the aid of the general identities,

$$\begin{aligned}
\left(\frac{\partial L}{\partial \dot{q}_j}\right)_q &= \left(\frac{\partial \tilde{L}}{\partial \dot{\theta}_i}\right)_q \left(\frac{\partial \dot{\theta}_i}{\partial \dot{q}_j}\right) \quad (i,j = 1,2,\ldots,n),\\
\left(\frac{\partial L}{\partial q_j}\right)_{\dot{q}} &= \left(\frac{\partial \tilde{L}}{\partial q_j}\right)_{\dot{\theta}} + \left(\frac{\partial \tilde{L}}{\partial \dot{\theta}_i}\right)_q \left(\frac{\partial \dot{\theta}_i}{\partial q_j}\right),\\
\delta\dot{\theta}_i &= \left(\frac{\partial \dot{\theta}_i}{\partial \dot{q}_j}\right)\delta\dot{q}_j + \left(\frac{\partial \dot{\theta}_i}{\partial q_j}\right)\delta q_j,
\end{aligned}$$

then

$$\delta L(q,\dot{q},t) = \delta\tilde{L}(q,\dot{\theta},t) = \left(\frac{\partial \tilde{L}}{\partial q_j}\right)\delta q_j + \left(\frac{\partial \tilde{L}}{\partial \dot{\theta}_i}\right)\delta\dot{\theta}_i \tag{7.9}$$

is invariant to any general $(q,\dot{q}) \Leftrightarrow (q,\dot{\theta})$-transformation. With the aid of (7.3)

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)_q \delta q_j = \left(\frac{\partial \tilde{L}}{\partial \dot{\theta}_i}\right)_q \left(\frac{\partial \dot{\theta}_i}{\partial \dot{q}_j}\right)_q \delta q_j = \left(\frac{\partial \tilde{L}}{\partial \dot{\theta}_i}\right)_\theta \delta\theta_i \tag{7.10}$$

is also invariant, but only for the particular linear transformation (7.1), where $\theta_i = \theta_i(q,t)$. The replacement,

$$\left(\frac{\partial \tilde{L}}{\partial q_j}\right)_{\dot{\theta}} \delta q_j = \left(\frac{\partial \tilde{L}}{\partial \theta_i}\right)_{\dot{\theta}} \left(\frac{\partial \theta_i(q,t)}{\partial q_j}\right)\delta q_j = \left(\frac{\partial \tilde{L}}{\partial \theta_i}\right)_{\dot{\theta}} \delta\theta_i,$$

can be made in (7.9), provided $(\partial\tilde{L}/\partial\theta_i)$ is identified as,

$$\left(\frac{\partial \tilde{L}}{\partial \theta_i}\right)_{\dot{\theta}} \overset{def}{\equiv} \left(\frac{\partial \tilde{L}(q,\dot{\theta},t)}{\partial q_j}\right)_{\dot{\theta}} \tilde{A}_{ji}\ , \tag{7.11}$$

obtained from (7.5). The transpositional form (4.25) of L can then be expressed in terms of $\tilde{L}$ as

$$L_j\delta q_j = \frac{d}{dt}\left[\left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\delta\theta_i\right] - \delta\tilde{L} + \left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)A_{ij}\left[\delta\dot{q}_j - \frac{d}{dt}\left(\delta q_j\right)\right] = Q_j^{NP}\delta q_j. \tag{7.12}$$

On subtracting (7.12) from the result,

$$\tilde{L}_i(\theta,\dot{\theta},t)\delta\theta_i = \frac{d}{dt}\left[\left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\delta\theta_i\right] - \delta\tilde{L} + \left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\left[\delta\dot{\theta}_i - \frac{d}{dt}\left(\delta\theta_i\right)\right], \tag{7.13}$$

of (4.25) applied to $\tilde{L}(\theta,\dot{\theta},t)$, then the transpositional form of the d'Alembert–Lagrange principle (2.2) in quasivelocities is obtained as

$$\boxed{\left(\tilde{L}_r - \tilde{Q}_r^{NP}\right)\delta\theta_r + \left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\left\{A_{ij}\left[\delta\dot{q}_j - \frac{d}{dt}\left(\delta q_j\right)\right] - \left[\delta\dot{\theta}_i - \frac{d}{dt}\left(\delta\theta_i\right)\right]\right\} = 0 \quad (i,j,r = 1,2,\ldots,n)}. \tag{7.14}$$

The relation $Q_j^{NP}\delta q_j = (Q_j^{NP}\tilde{A}_{ji})\,\delta\theta_i \equiv \tilde{Q}_i^{NP}\delta\theta_i$ determines the transformed generalized force $\tilde{Q}_i^{NP}$ to be $Q_j^{NP}\tilde{A}_{ji}$. With the aid of the primary rule (7.6), the quasi-velocity form of DLP is

$$\left[\tilde{L}_r - \tilde{Q}_r^{NP} + \left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\dot{\theta}_{ij}\tilde{A}_{jr}\right]\delta\theta_r = 0 \quad (i,j,r = 1,2,\ldots,n)\,. \tag{7.15}$$

When the quasivelocities and constraints are integrable, $\dot{\theta}_{ij}\delta q_j = \dot{\theta}_{ij}\tilde{A}_{jr}\ \delta\theta_r = 0$, the coordinates are true and (7.14), with the aid of (7.5), reduces, as expected, to the original d'Alembert–Lagrange equation (2.2). The solution (7.4) for $\dot{q}_j$ is used to express L, A_{sj} and $\dot{\theta}_{ij}$, the Lagrangian derivatives (2.16) of (7.1), in terms of $(q,\dot{\theta})$.

The only other direct derivation[49] of (7.15) from transpositional relations assumed from the beginning that all (dependent and independent) velocity displacements obeyed the traditional commutation rule (2.14) for $\delta\dot{q}_j$ to give,

$$\left(\tilde{L}_r - \tilde{Q}_r^{NP}\right)\delta\theta_r - \left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\left[\delta\dot{\theta}_i - \frac{d}{dt}\left(\delta\theta_i\right)\right] = 0, \quad (i,r = 1,2,\ldots,n). \tag{7.16}$$

Then (7.16), with the aid of (2.14) in (7.6), yields (7.15). The more general relation (7.6) however reduces (7.15) to (7.14) so that assumption (2.14) which precludes possible displaced states for $i = m, m+1,\ldots,n$ is not required.

B. Boltzmann–Hamel equation for linear-velocity constraints

Under constraints, the first m displacements $\delta\theta_r$ are then all independent and the remaining c displacements $\delta\theta_s$ are all zero, so that (7.15) yields the minimum number m of equations of state

$$\tilde{L}_r + \left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\dot{\theta}_{ij}\tilde{A}_{jr} = \tilde{Q}_r^{NP} \quad (r = 1,2,\ldots,m) \quad (i,j = 1,2,\ldots,n)\,, \tag{7.17}$$

where the linear-velocity constraints (2.11) are embedded. It is therefore free from Lagrange's multipliers and involves the unconstrained Lagrangian until completion of the $\dot{\theta}_i$-differentiation for $i > m$ of $\tilde{L}(q;\dot{\theta}_1,\dot{\theta}_2,\ldots,\dot{\theta}_r,\ldots,\dot{\theta}_n,t)$ in (7.17), $\tilde{L}_r$ remaining unaffected when the constrained $\tilde{L}$ is used. The $\dot{\theta}_{ij}$ are the Lagrangian derivatives (2.16) of all the quasivelocities (7.1), including the constraints. On using (2.16) for $\dot{\theta}_{ij}$, (7.17) reproduces the standard form[1,14,16,49] of the Boltzmann–Hamel equation

$$\tilde{L}_r + \left(\frac{\partial\tilde{L}}{\partial\dot{\theta}_i}\right)\left[\gamma_{rs}^i\dot{\theta}_s + \gamma_r^i\right] = \tilde{Q}_r^{NP} \quad (r,s = 1,2,\ldots,m) \quad (i = 1,2,\ldots,n), \tag{7.18}$$

where index r labels each of the m equations written in terms of the Hamel coefficients,[14]

$$\gamma_{rs}^i = \left(\frac{\partial A_{ij}}{\partial q_k} - \frac{\partial A_{ik}}{\partial q_j}\right)\tilde{A}_{jr}\tilde{A}_{ks} = -\gamma_{sr}^i \quad (i,j,k,l = 1,2,\ldots,n), \tag{7.19}$$

$$\gamma_r^i = -\left(\frac{\partial A_{ij}}{\partial q_k} - \frac{\partial A_{ik}}{\partial q_j}\right)\tilde{A}_{jr}(\tilde{A}_{kl}B_l) + \left(\frac{\partial A_{ij}}{\partial t} - \frac{\partial B_i}{\partial q_j}\right)\tilde{A}_{jr}, \tag{7.20}$$

which involve double and treble summations over (j,k) and (j,k,l), respectively. In the absence of constraints, (7.17) reduces to the pure quasivelocity form (7.18), where the indices are now $(r,s = 1,2,\ldots,n)$. Beginning with (7.18), Greenwood[16] assumed the commutation rule for $\delta\dot{\theta}_j$ and obtained,

$$\left(\tilde{L}_r - \tilde{Q}_r^{NP}\right)\delta\theta_r + \left(\frac{\partial \tilde{L}}{\partial \dot{\theta}_i}\right)A_{ij}\left[\delta\dot{q}_j - \frac{d}{dt}\left(\delta q_j\right)\right] = 0, \quad (i,j,r = 1,2,\ldots,n)\,. \tag{7.21}$$

Our present development has shown that the transpositional form of the Boltzmann-Hamel equation (7.18) is (7.14), which is more general than the previous forms (7.16) and (7.21).

VIII. SUMMARY AND CONCLUSION

We have shown from basic principles how the d'Alembert–Lagrange principle (2.2) can be implemented for general nonholonomic systems to provide, in a quite natural fashion, equations of state (3.16) and (3.25) for general velocity and acceleration constraints (3.1) and (3.2), respectively. These equations are correct because they agree with the EOS (A15) and (A16) derived in Appendix A from the quite different principle of Gauss. From the property of possible displaced states, implementation of DLP was effected by establishing sets (3.15) and (3.24) of linear conditions which restrict the displacements δq_j. These sets facilitated construction of new transpositional relations for nonholonomic systems. For velocity constraints (3.1), the set (3.15) implies

$$\delta g_k - \frac{d}{dt}\left[\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j\right] = 0 \quad (k = 1,2,\ldots,c)\,, \tag{8.1}$$

which, when inserted into the basic transpositional relation (4.3) provides the new transpositional relation,

$$\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta\dot{q}_j - \frac{d}{dt}(\delta q_j)\right] = g_{kj}\delta q_j \quad (j = 1,2,\ldots,n)\,, \tag{8.2}$$

from which the velocity displacements can be determined from the configuration displacements. It is noted that the q-space commutation rule (2.14) traditionally accepted in Lagrangian dynamics for dependent and independent q precludes possible nonholonomic displaced states and prohibits any further advance of nonholonomic theory. For integrable systems, $g_{kj}\delta q_j = 0$ and (8.2) reduces to the traditional commutation rule (2.14). On taking the independent coordinates to satisfy (2.14), (8.2) was solved to provide subrules (4.13).

Analogous sets of relations (3.26), (5.3), and (5.5) were also established for nonholonomic systems under acceleration constraints (3.2). Various hierarchies of higher-order transpositional relations were then constructed for nonholonomic systems. They reproduce the lower-order results in the appropriate limits and elucidate various interconnections. The reverse procedure of invoking these commutation relations (8.2) and (5.3) to furnish the EOS (3.16) and (3.25) is also viable.

The present work shows that (7.13) is the most general transpositional form of the Boltzmann–Hamel equation. Also shown is that (8.2) does not affect the transpositional derivation of the Boltzmann–Hamel equation. It affects, however, Hamilton's integral principle (6.3) which does not now hold for nonintegrable systems, in contrast to previous conclusions[6,15] based on (2.14). Hamilton's principle (6.5) of least action maintains its validity only for integrable constraints, irrespective of either transpositional relation, (8.2) or (2.14). Vakonomic mechanics was also placed in context of the present work and provides results which agree with those of DLP *only* for integrable velocity constraints. In the course of this work, the Chetaev rule,[37] which had been previously left unjustified from basic principles, was examined as an aside in Appendix A. The rule is apparently validated by the present proof of (3.15) and (3.24).

In conclusion, the long-standing problem on the implementation of the d'Alembert–Lagrange principle to dynamical systems under nonlinear kinematic constraints has been solved and the correct equations of state (3.16) and (3.25) have been derived from (2.2). Useful transpositional relations for nonholonomic systems have also been established.

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APPENDIX A: NONHOLONOMIC EQUATIONS OF STATE DERIVED FROM THE GAUSS PRINCIPLE

Because textbooks have provided no comprehensive account or common source, we present a unified derivation of the equations of state for nonholonomic systems under general velocity and acceleration constraints from the Gauss variational principle, a principle quite different from the d'Alembert–Lagrange principle. The derivation, based on the work of Gibbs,[23] Appell,[26,27] and Ray,[28] yields results which confirm the correctness of our present EOS (3.16) and (3.25) obtained from DLP. The Gauss "constraint,"

$$C(\ddot{\mathbf{r}}|\dot{\mathbf{r}},\mathbf{r},t) = \frac{1}{2}m_i\left(\frac{\mathbf{F}_i^C}{m_i}\right)^2 = \left(\frac{1}{2m_i}\right)(m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i)^2\,, \tag{A1}$$

is the "kinetic energy of acceleration" due to the constraint forces $\mathbf{F}_i^C$ alone. It also provides the difference $(C/2m_i)^{1/2}\Delta t^2$ between the final constrained and constraint-less configurations of the system evolving after time interval Δt from the common fixed initial state $\{\mathbf{r}(t),\dot{\mathbf{r}}(t)\}$. Let $\Delta_2\ddot{\mathbf{r}}_i$ be a finite variation only to the acceleration $\ddot{\mathbf{r}}_i$, keeping the physical state $(\mathbf{r}_i,\dot{\mathbf{r}}_i)$ of all particles with mass m_i fixed at time t. The external force $\mathbf{F}_i(\mathbf{r},\dot{\mathbf{r}},t)$ on each particle i is therefore unaffected by this type of variation. The finite change in C is exactly

$$\Delta_2 C = \mathbf{F}_i^C.\Delta_2\ddot{\mathbf{r}}_i + \frac{1}{2}m_i(\Delta_2\ddot{\mathbf{r}}_i)^2\,. \tag{A2}$$

For infinitesimal displacements, $\Delta_2 \to \delta_2$ so that the first-order infinitesimal change is

$$\delta_2 C = \mathbf{F}_i^C.\delta_2\ddot{\mathbf{r}}_i = \nabla_{\ddot{\mathbf{r}}_i}\left(\frac{1}{2}m_i\ddot{\mathbf{r}}_i^2 - \mathbf{F}_i.\ddot{\mathbf{r}}_i\right).\delta_2\ddot{\mathbf{r}}_i\,. \tag{A3}$$

On assuming that the unknown constraint forces obey $\mathbf{F}_i^C.\delta_2\ddot{\mathbf{r}}_i = 0$, then C is a minimum, because the second-order change in C is the positive quadratic term $\delta_2 C = \frac{1}{2}m_i(\delta_2\ddot{\mathbf{r}}_i)^2 \geq 0$. The Gauss principle[14,15,20,21]

$$\left(\nabla_{\ddot{\mathbf{r}}_i}C\right).\delta_2\ddot{\mathbf{r}}_i = 0 \quad \text{(Gauss principle)} \tag{A4}$$

is then obtained. C is stationary with respect to virtual infinitesimal variations to the acceleration alone. For particles free from constraints, C decreases to its absolute minimum of zero where Newton's second law is recovered for each particle. In general, the motion is such that C due to constraints is least. In contrast to the original d'Alembert–Lagrange *differential*-variational principle (2.2), (A4) is a true minimum principle. It is *on a par* with Hamilton's variational principle of least action for holonomic constraints, but with the additional and powerful advantage that it can also be applied to general nonholonomic constraints.

Both classes (3.1) and (3.2) of kinematic constraints can now be treated by the generalized-coordinate version of (A4). Here, the Newtonian variation $\delta_2\ddot{\mathbf{r}}_i$ is $(\partial\ddot{\mathbf{r}}_i/\partial\ddot{q}_j)\delta_2\ddot{q}_j = (\partial\mathbf{r}_i/\partial q_j)\delta_2\ddot{q}_j$, in terms of the variation $\delta_2\ddot{q}_j$ *only* to acceleration $\ddot{q}_j$ associated with the fixed physical state $\{q,\dot{q}\}$ of the system. The generalized-coordinate version of (A4) is then the variation,

$$\delta_2 R = \left(\frac{\partial R}{\partial \ddot{q}_j}\right)\delta_2\ddot{q}_j = 0 \quad \text{(Gauss principle)}\,, \tag{A5}$$

of

$$R(\ddot{q}|\dot{q},q,t) \overset{def}{\equiv} \frac{1}{2}m_i\ddot{\mathbf{r}}_i^2(\ddot{q}|\dot{q},q,t) - Q_j(q,\dot{q},t)\ddot{q}_j \tag{A6}$$

with respect to $\ddot{q}$ alone, where $\frac{1}{2}m_i\ddot{\mathbf{r}}_i^2$ is the Gibbs' function or full "kinetic energy of acceleration." For a constraint-free system, $\delta_2\ddot{q}_j$ are independent so that (A5) provides the Gibbs–Appell equations[14–16,23,26–29]

$$\frac{\partial}{\partial\ddot{q}_j}\left(\frac{1}{2}m_i\ddot{\mathbf{r}}_i^2\right) = Q_j \quad (j = 1,2,\ldots n)\,. \tag{A7}$$

An important note is that (A7), with the identity (A15) given below, reduces to Lagrange's equations (2.4). When quasivelocities (7.1) are adopted for linear-velocity systems, the form of (A7) remains invariant[14,15] and reduces to the minimal set of m equations, free from Lagrange's multipliers. Lagrange's equations, on the other hand, transform to the much more awkward form[14,16] of the Boltzmann–Hamel equation (7.19).

Under general acceleration constraints (3.2), a finite virtual acceleration displacement $\Delta_2\ddot{q}$ to the actual acceleration $\ddot{q}$ of a physical state $(q,\dot{q})$ to a new displaced "state" $(q,\dot{q},\ddot{q}+\Delta_2\ddot{q})$ at time t is kinetically possible provided the constraint equations

$$h_k(\ddot{q}+\Delta_2\ddot{q}|\dot{q},q,t) = \left(\frac{\partial h_k}{\partial\ddot{q}_i}\right)\Delta_2\ddot{q}_i + \frac{1}{2!}\left(\frac{\partial^2 h_k}{\partial\ddot{q}_i\partial\ddot{q}_j}\right)(\Delta_2\ddot{q}_i)(\Delta_2\ddot{q}_j) + \ldots.. = 0$$

are satisfied at the displaced state. For linear acceleration constraints (4.27), only the first RHS-term remains so that

$$\Delta_2 h_k^{(1)} = h_k^{(1)}(\ddot{q}+\Delta_2\ddot{q}|\dot{q},q,t) = \left(\frac{\partial g_k}{\partial\dot{q}_j}\right)\ \Delta_2\ddot{q}_j = 0, \tag{A8}$$

where $\Delta_2\ddot{q}_j$ can remain finite. For general acceleration constraints (3.2), we must now take the displacements as the infinitesimal quantities $\delta_2\ddot{q}_i$ because they will then provide the set

$$\delta_2 h_k = h_k(\ddot{q}+\delta_2\ddot{q}|\dot{q},q,t) = \left(\frac{\partial h_k}{\partial\ddot{q}_j}\right)\ \delta_2\ddot{q}_j = 0 \tag{A9}$$

of linear conditions which can be adjoined to (A5) to give the constrained Gauss principle[27]

$$\frac{\partial\tilde{R}}{\partial\ddot{q}_j} = 0 \quad (\text{constrained Gauss}'\text{ principle}), \tag{A10}$$

where the constrained function $\tilde{R}$ is $(R - \lambda_k h_k)$ and where the $\delta_2\ddot{q}_j$ are now all regarded as independent. The equations of state are therefore

$$\frac{\partial R}{\partial\ddot{q}_j} = \lambda_k\frac{\partial h_k}{\partial\ddot{q}_j} \quad (j = 1,2,\ldots,n) \tag{A11}$$

or equivalently

$$\frac{\partial}{\partial\ddot{q}_j}\left(\frac{1}{2}m_i\ddot{\mathbf{r}}_i^2\right) = Q_j + \lambda_k\frac{\partial h_k}{\partial\ddot{q}_j} \quad (j = 1,2,\ldots n)\,. \tag{A12}$$

These equations, first given by Appell,[27] extend the standard Gibbs–Appell equations (A7) to nonlinear velocity and acceleration constraints. Calculation of the Gibbs acceleration function $(\frac{1}{2}m_i\ddot{\mathbf{r}}_i^2)$ and then its $\ddot{q}_j$-derivatives are however more complicated than the much simpler Lagrangian derivative L_j. Direct application of the Gibbs–Appell equation (A13) then becomes more labor intensive than (3.16) and (3.25), which involve only the (simpler-to-calculate) Lagrangian. However, the transformation between Lagrangian and Newtonian formulations is facilitated by the useful identity

$$L_j - Q_j^{NP} = (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i).\left[\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right) = \left(\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j}\right) = \left(\frac{\partial\ddot{\mathbf{r}}_i}{\partial\ddot{q}_j}\right)\right], \tag{A13}$$

where the generalized force $Q_j = \mathbf{F}_i.(\partial\mathbf{r}_i/\partial q_j)$ is the sum of a potential part, absorbed within the Lagrangian derivative L_j, and a nonpotential part Q_j^{NP}. The Lagrangian form of (A12) is then

$$L_j - Q_j^{NP} = \frac{\partial}{\partial\ddot{q}_j}\left[\frac{1}{2}m_i\ddot{\mathbf{r}}_i^2(\ddot{q}|\dot{q},q,t) - Q_j(q,\dot{q},t)\ddot{q}_j\right] = \lambda_k\frac{\partial h_k}{\partial\ddot{q}_j} \quad (j = 1,2,\ldots n)\,. \tag{A14}$$

Identity (A13) illustrates that $(L_j - Q_j^{NP})\,\delta q_j$ is the sum of all projections of Newton's equations, without constraints, onto the surfaces $\mathbf{r}_i(q,t)$ for all particles i. Ideal constraints, defined as $Q_j^C\delta q_j = \mathbf{F}_i^C.\delta\mathbf{r_i} = 0$, are therefore normal to these surfaces. Then (A14) provides the (simpler-to-apply) Lagrangian forms

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \lambda_k\left(\frac{\partial g_k}{\partial \dot{q}_j}\right) \quad (j = 1,2,\ldots,n)} \tag{A15}$$

and

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NP} + \lambda_k\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right) \quad (j = 1,2,\ldots,n)} \tag{A16}$$

of (A12) for the equations of state for nonholonomic systems under general velocity and acceleration constraints (3.1) and (3.2), respectively. These are in exact agreement with the EOS (3.16) and (3.25) derived in the text from the d'Alembert–Lagrange principle (2.2). An important consequence of (A15) is that the Gauss principle (A5) can now be expressed in Lagrangian form as

$$\boxed{\left(L_j - Q_j^{NP}\right)\,\delta_2\ddot{q}_j = 0 \quad \text{(Lagrangian form of Gauss principle)}}, \tag{A17}$$

solved subject to auxiliary conditions (A8) and (A9) which ensure that the displaced states are all possible. The form (A17) was first noted by Gibbs[23] for systems under holonomic constraints.

Until the present analysis, (A15) and (A16) have eluded direct derivation from the d'Alembert–Lagrange principle (2.2). Their basis (A17) is however analogous to (2.2), where the configuration displacement δq_j is replaced by the acceleration displacement $\delta_2\ddot{q}_j$ for velocity constraints and nonlinear acceleration displacements, to be solved subject to conditions (A8) and (A9) on the displacements. This analogy therefore raises the following correspondence.

1. The Chetaev rule

On comparing (A17) with (2.2), it is tempting to suggest that (A15) and (A16) could well be obtained more easily, without all the details involved in the above application of the Gauss principle (A4), simply by asserting, without proof, that the δq_j displacements in the d'Alembert–Lagrange principle (2.2) should obey the prescriptions,[37]

$$\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\,\delta q_j = 0, \tag{A18}$$

for velocity constraints and[44]

$$\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\,\delta q_j = 0 \tag{A19}$$

for acceleration constraints. These are the *ad hoc* rules proposed[37,38,44] for kinematic constraints (3.1) or (3.2). Although the Chetaev rule (A18) is the DLP-condition (2.12) for linear constraints (2.11) and agrees with the present result (3.5) for homogeneous velocity constraints, it has remained without theoretical validation for general g_k. Correspondence with the result (A15) of the different Gauss principle also suggests its acceptance. Although the solutions (A15) of $L_j\delta_2\ddot{q}_j = 0$ subject to (A8) and of $L_j\delta q_j = 0$ subject to (A18) both coincide, the restrictions (A8) and (A18) imposed on their respective displacements are however quite different—$\delta_2\ddot{q}_j$ keeps the state $(q,\dot{q})$ fixed, whereas δq_j changes the state. Similarly, the solution (A16) is based on condition (A9), which is quite different from (A19). Although generally accepted as an axiom in nonholonomic dynamics, (A18) has never been theoretically justified or proven directly from DLP which is based on displacements δq_j to the configuration, which in turn cause displacements in all q, $\dot{q}$, and $\ddot{q}$. Although many attempts[28,38,43–46] have been made to reconcile (A18) with DLP, mainly by seeking suitable forms for g_k which satisfy (A18), as, for example, homogeneous velocity constraints, the rule has been generally accepted

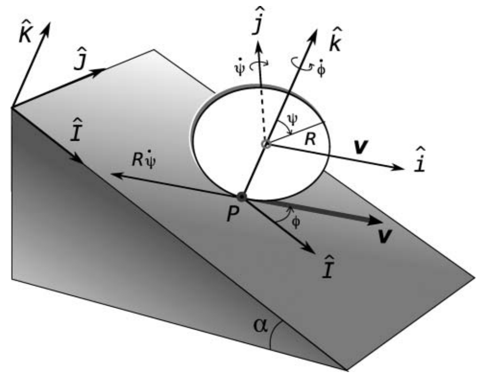

FIG. 2. The penny rolls upright while turning on an inclined plane of angle α. Directions of space-fixed axes are $\hat{I}$, $\hat{J}$, and $\hat{K}$, as indicated. Disk rolls along the plane with angular velocity $\dot{\psi}\hat{j}$ about symmetry axis $\hat{j}(t)$ which turns with constant angular velocity $\dot{\phi}\hat{k}$ about fixed figure axis $\hat{k}$. The CM has velocity $\mathbf{v}(t) = [R\dot{\psi}(t)]\hat{i}(t)$ and the point of contact P is instantaneously at rest.

into the methodology of analytical dynamics. Because there was apparently no way to derive (A18) directly from (3.1), (A18) has remained unjustified and therefore a contentious issue.[9]

This long-standing enigma has now been resolved in the text by analysis based on possible displaced states which provides the sets (3.15) and (3.24) of linear conditions on the displacements δq_j required for the successful implementation of the d'Alembert–Lagrange principle to nonholonomic systems under general kinematic constraints. The procedure also leads, in a natural fashion, to a set of transpositional relations, which may, in turn, be invoked *ab initio* to rederive the EOS. A consequence of the present development is that the axioms (A18) and (A19), previously unjustified, are now validated by the explicit proofs of the identical conditions (3.15) and (3.24) presented in the text.

APPENDIX B: THE TWO TRANSPOSITIONAL RELATIONS

Options 1 and 2 of Sec. IV A are based on transpositional relations (2.14) and (4.6), respectively. The disadvantage of option 1 is that displaced states are not possible for nonintegrable constraints and any theoretical advance beyond (4.4) is inhibited. The advantage of (4.6) over (2.14) is that the δq displacement causes transition to a possible state, and in doing so provides the set (3.15) of linear restrictions on the δq_j required for implementation of DLP. The sum $g_{ki}\delta q_i$ will vanish only for integrable velocity constraints (Sec. IV A 2), so that the traditional commutation relation (2.14) is recovered, in this limit, from (4.6). Both options can be exercised for only holonomic and linear-velocity constraints. Option 2 provides the only viable method for general nonholonomic constraints.

1. Example of nonintegrable linear-velocity constraints: The nonholonomic penny

Example 4: The two different approaches based on (2.14) and on (4.6) or (4.8) are now illustrated via a specific example of nonintegrable linear-velocity constraints. The familiar nonintegrable constraints for the nonholonomic upright penny of radius R, rolling with speed $R\dot{\psi}$ and turning with angular speed $\dot{\phi}$ on the inclined plane of Fig. 2, are,

$$\dot{\theta}_1 = G_1 = \dot{x} - R\dot{\psi}\cos\phi = 0, \tag{B1}$$

$$\dot{\theta}_2 = G_2 = \dot{y} - R\dot{\psi}\sin\phi = 0, \tag{B2}$$

the Cartesian components of $\mathbf{v}_P$, where (x, y) are the Cartesian coordinates of the point of contact P with the plane, and where ϕ is the angle between the x axis and the penny's velocity along the tangent to the curve $x(t), y(t)$. Let $\dot{x}, \dot{y}$ be the dependent velocities. The coordinate functions $\theta_k(q, t)$

are unknown for nonintegrable constraints, but the displacements obey the conditions

$$\delta\theta_1 = \delta x - (R\cos\phi)\delta\psi = 0, \tag{B3}$$

$$\delta\theta_2 = \delta y - (R\sin\phi)\delta\psi = 0, \tag{B4}$$

which follow directly from (B1) and (B2), or equivalently from the general formula (2.12) for linear-velocity constraints. At the displaced state, the constraints change from (B1) and (B2) to

$$\delta\dot{\theta}_1 = \delta G_1 = \delta\dot{x} - (R\cos\phi)\delta\dot{\psi} + (R\sin\phi)\dot{\psi}\delta\phi, \tag{B5}$$

$$\delta\dot{\theta}_2 = \delta G_2 = \delta\dot{y} - (R\sin\phi)\delta\dot{\psi} - (R\cos\phi)\dot{\psi}\delta\phi\ . \tag{B6}$$

Method 1: This is characterized by the commutation rule (2.14) and the general relation (4.4) which takes the form (4.5) for the linear-velocity constraints (B1) and (B2). Displaced states are not possible. Application of (2.14) on (B3) and (B4) yields

$$\frac{d}{dt}(\delta\theta_1) = \delta\dot{x} - (R\cos\phi)\delta\dot{\psi} + (R\sin\phi)\dot{\phi}\delta\psi = 0, \tag{B7}$$

$$\frac{d}{dt}(\delta\theta_2) = \delta\dot{y} - (R\sin\phi)\delta\dot{\psi} - (R\cos\phi)\dot{\phi}\delta\psi = 0\ . \tag{B8}$$

The differences

$$\delta\dot{\theta}_1 - \frac{d}{dt}(\delta\theta_1) = \delta G_1 = -R\sin\phi\left(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi\right) \tag{B9}$$

$$\delta\dot{\theta}_2 - \frac{d}{dt}(\delta\theta_2) = \delta G_2 = R\cos\phi\left(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi\right) \tag{B10}$$

are then obtained from (B5)–(B8). The only nonzero Lagrangian derivatives calculated from (4.2) are $G_{1\psi} = R\dot{\phi}\sin\phi$, $G_{1\phi} = -R\dot{\psi}\sin\phi$, $G_{2\psi} = -R\dot{\phi}\cos\phi$, and $G_{2\phi} = R\dot{\psi}\cos\phi$. Application of formula (4.5) then provides results identical with (B9) and (B10).

Method 2: This is characterized by (4.6), which, for linear-velocity constraints, has the commutation form (4.8) in θ-space. On using (B3)–(B6) and the condition $\delta\dot{\theta}_{1,2} = \delta G_{1,2} = 0$ for possible states, the commutation relation (4.8) then provides two transpositional relations

$$\left[\delta\dot{x} - \frac{d}{dt}(\delta x)\right] - R\cos\phi\left[\delta\dot{\psi} - \frac{d}{dt}(\delta\psi)\right] = +R\sin\phi\left(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi\right), \tag{B11}$$

$$\left[\delta\dot{y} - \frac{d}{dt}(\delta y)\right] - R\sin\phi\left[\delta\dot{\psi} - \frac{d}{dt}(\delta\psi)\right] = -R\cos\phi\left(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi\right), \tag{B12}$$

which link the two (dependent) velocity displacements $\delta\dot{x}$ and $\delta\dot{y}$ to the free arbitrary displacements and which highlight the fact that $\delta\dot{x}$ and $\delta\dot{y}$ cannot satisfy commutation (2.14). Application of formula (4.6) yields results identical with (B11) and (B12), whose RHSs are simply $G_{kj}\delta q_j$. The displaced states are possible in method 2.

Four equations are required to specify the transpositional rules for the four coordinates (x, y, ϕ, ψ). We may therefore allow the two independent and arbitrary coordinates (ψ,ϕ) in (B11) and (B12) to satisfy commutation rule (2.14). The dependent velocity displacements then obey the transpositional relations

$$\left[\delta\dot{x} - \frac{d}{dt}(\delta x)\right] = R\sin\phi\left(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi\right) \tag{B13}$$

$$\left[\delta\dot{y} - \frac{d}{dt}(\delta y)\right] = R\cos\phi\left(\dot{\psi}\delta\phi - \dot{\phi}\delta\psi\right) \tag{B14}$$

in agreement with the established subrules (4.13). When the constraints are rewritten in the equivalent form

$$g_1 = \dot{x}\cos\phi + \dot{y}\sin\phi - R\dot{\psi} = 0, \tag{B15}$$

$$g_2 = \dot{x}\sin\phi - \dot{y}\cos\phi = 0, \tag{B16}$$

the tangential and normal components of $\mathbf{v}_P$, the rule (4.6) yields

$$\left[\delta\dot{x} - \frac{d}{dt}(\delta x)\right]\cos\phi + \left[\delta\dot{y} - \frac{d}{dt}(\delta y)\right]\sin\phi - R\left[\delta\dot{\psi} - \frac{d}{dt}(\delta\psi)\right] = 0, \tag{B17}$$

$$\left[\delta\dot{x} - \frac{d}{dt}(\delta x)\right]\sin\phi - \left[\delta\dot{y} - \frac{d}{dt}(\delta y)\right]\cos\phi = R\left(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi\right), \tag{B18}$$

which are simple linear combinations of the set (B11) and (B12). Thus application of (4.6) to constraints, written in different equivalent forms, (B1) and (B2) or (B15) and (B16), produce equivalent results. The full solution $x(t), y(t), \phi(t), \psi(t)$ of (3.16) and the interesting geometrical paths obtained for the nonholonomic penny are provided elsewhere.[52]

In summary, method 1 is based on commutation relation (2.14) in q-space and the resulting relation (4.4) which implies that the displaced states violate the constraints. Method 2 is based on the noncommuting relation (4.6) which allows possible displaced states for kinematic constraints. Although derivation of EOS for homogeneous velocity constraints in Sec. III A does not rely on either method, method 2 possesses a clear advantage for general kinematic constraints, because it is open to the property that displaced states are possible, thereby allowing the condition (3.15) to be extracted for the implementation of DLP to obtain the correct EOS (3.16).

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